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THE KERNEL OF MONOID MORPHISMS

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Introduction

This paper introduces what the authors believe to be the correct definition of the kernel of a monoid morphism $\varphi : M \to N$. This kernel is a category, constructed directly from the constituents of φ. In the case of a group morphism, our kernel is a groupoid that is divisionally equivalent to the traditional kernel.

This article is a continuation of the work in [8]. The thesis of [8] is that categories, as generalized monoids, are essential ingredients in monoid decomposition theory. The principal development in [8] was the introduction of *division*, a new ordering for categories, which extend the existing notion for monoids. Since its introduction in [3], division has proved to be the ordering of choice for monoids. This extension of division to categories allows for the useful comparison of monoids and categories.

A strong candidate for the title 'kernel' was introduced in [8]. This candidate is also a category and is called the derived category of φ. The derived category operation and the wreath product of monoids are shown to have an adjoint-like relationship. This relationship is summed up in the Derived Category Theorem [8, Theorem 5.2]. The derived category has its origins in [6], where it appears as the derived semigroup.

The kernel construction of this paper is an improvement over the derived category for a variety of reasons. First, it is smaller in the divisional sense. Second, it is a reversal invariant construction. Third, it combines more effectively with classical structure theories. For example, when applied to surmorphisms that cannot be further factored, the kernel has a particularly simple form. This leads to important decomposition theorems for finite monoids.

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A new product, the *block* product, is introduced to complement the kernel construction. The kernel and block product have the same adjoint-like relationship as the derived category and the wreath product. The wreath product is a specific form of the semidirect product; the block product is derived similarly from a two-sided semidirect product, called a double semidirect product.

The block product is an improvement over the wreath product for the following reasons. First, it is larger in the divisional sense. Second, it is a reversal invariant construction. Third, it admits a prime decomposition theorem for finite monoids that is an improvement over the wreath product version of [3].

The paper may be logically divided into two parts, based on cardinality. Sections 1, 5–7 deal with arbitrary monoids and categories. Sections 2–4 treat finite monoids and categories.

The kernel of a relation $\varphi: M \to N$ of monoids is introduced and its basic properties developed in Section 1. A relation of monoids is a relation whose graph $\#\varphi = \{(m,n): n \in m\varphi\}$ is a submonoid of $M \times N$. This concept includes morphism and division.

The kernel provides the foundation for a prime decomposition theorem of finite relations of monoids. This is the subject of Sections 2–4. The result (Theorem 3.1) states that every relation may be written as a composition of 'primitive' relations. A relation is primitive if its kernel bears a certain relationship to a simple monoid, that is, a monoid with no non-trivial homomorphic images. The finite simple monoids consist of the simple groups and the two element monoid $U_1 = \{1, 0\}$.

Section 2 introduces primitive relations and the notion of $\mathscr{P}$-free relations. Section 3 states the central theorem, presents corollaries, and reduces the proof of the theorem to the case of maximal proper surjective morphisms (MPS). Section 4 utilizes the classification of MPS's that appear in [5] to prove the theorem. This proof is a synthesis of the classical structure theories of finite monoids, and the more recent category developments of [8].

Starting with Section 5, we again treat arbitrary monoids. Section 5 develops addition properties of the kernel. Section 6 introduces the double semidirect product, which generalizes the semidirect, reverse semidirect, and triple products. Theorem 6.2 establishes the deep connection between the kernel and the double semidirect product.

Section 7 introduces the block product, which is a specific instance of a double semidirect product. The Kernel Theorem (Theorem 7.4) states the connection between the kernel and the block product. Theorems 6.2 and 7.4 combine to establish an adjoint-like relationship between these two concepts.

The main body of this paper is written with the assumption that certain (standard) facts about finite monoids and semigroups are understood. It is also assumed in the body of the text that the concepts of division and relational morphism are understood. Two appendices are provided with additional background information and exposition to fill this gap.

An equivalence relation on categories is used here and in [8] that differs from the 'natural equivalence' of standard category theory. Our equivalence derives from division. We say categories are (divisionally) equivalent (and write $S \sim T$) if they divide each other. Divisional equivalence is broader than natural equivalence. Please see [8, Section 3] for a discussion of this subject. For a good category theory source, MacLane's [4] is recommended.

A few comments about notation used in this paper need to be made. Categories are treated here as algebraic objects rather than classifying tools for mathematical structures. For this reason, the term 'morphism' is not used as a synonym for 'arrow', but is synonymous with the term 'functor'. In other words, a morphism is an arrow in the category of categories and functors. Another variant is the notation $S(c, c')$, which means the hom-set $\mathrm{Hom}_S(c, c')$ of category S. This notation is used to suggest an algebraic setting and to reduce subscripting.

Throughout this paper we will be treating functions of two variables, and all the functions to be encountered will have a property which could be called 'acting in the middle'. We, therefore, adopt the following notation. Let

$$F: X \times Y \to Z$$

be a function of two variables. Rather than writing $(x, y)F$ or $F(x, y)$ for the value obtained by applying F to (x, y), we shall use the notation xFy. This notation enjoys the property of being parenthesis-free. It also allows for a natural way of expressing the essential ideas of the kernel, the double semidirect product and the block product.

Due to length considerations, this is Part One of a two part paper. Part Two applies the results of this paper in a variety setting.

1. The kernel

Let M and N be monoids. A relation of monoids $\varphi: M \to N$ is a set relation with the property that its *graph*

$$\#\varphi = \{(m, n): n \in m\varphi\}$$

is a submonoid of $M \times N$. Equivalently, $\varphi: M \to N$ is a relation of monoids if

$$1 \in 1\varphi \quad \text{and} \quad m\varphi m'\varphi \subseteq (mm')\varphi$$

for all $m, m' \in M$. Evidently, the inverse of a relation of monoids is also a relation of monoids. Morphisms and divisions of monoids are relations of monoids.

Let $\varphi: M \to N$ be a relation of monoids. We construct a category K_φ, which we call the *kernel* of φ. This construction is best described in two steps: First, a category W_φ is constructed. Second, a congruence η on W_φ is constructed. The kernel of φ will be the quotient W_φ/η, and the associated quotient morphism will be denoted $\eta: W_\varphi \to K_\varphi$.

The objects of W_φ (and hence K_φ) are pairs of elements from the image, $M\varphi$, of φ. That is,

$$\mathrm{Obj}(W_\varphi)=\{(n,n'): n,n'\in M\varphi\}.$$

Because the objects are pairs, we find it convenient to adopt the following notational conventions. When Y is a set, a boldface $\mathbf{Y}$ will denote the direct product $Y\times Y$. A boldface $\mathbf{y}$ will denote a pair $(y_\mathrm{L},y_\mathrm{R})\in \mathbf{Y}$; for example, we have $\mathbf{y}'=(y'_\mathrm{L},y'_\mathrm{R})$. The subscripts 'L' and 'R' will be reserved for this purpose.

The arrows of W_φ are derived from the elements of $\#\varphi$. A pair $(m,n)\in\#\varphi$ defines an arrow

$$(1.1)\qquad (m,n):\mathbf{n}\to\mathbf{n}'$$

if $n_\mathrm{L}n=n'_\mathrm{L}$ and $n_\mathrm{R}=nn'_\mathrm{R}$. Composition is that of $\#\varphi$. That is, given the arrow

$$(1.2)\qquad (m',n'):\mathbf{n}'\to\mathbf{n}'',$$

the product of (1.1) and (1.2) is the arrow

$$(mm',nn'):\mathbf{n}\to\mathbf{n}''.$$

Since $n_\mathrm{L}n=n'_\mathrm{L}$ and $n'_\mathrm{L}n'=n''_\mathrm{L}$, we see that $n_\mathrm{L}nn'=n''_\mathrm{L}$. Similarly, $n_\mathrm{R}=nn'n''_\mathrm{R}$. Therefore, this composition rule is well defined and associative. Note also that for each $\mathbf{n}\in\mathrm{Obj}(W_\varphi)$, the arrow $(1,1):\mathbf{n}\to\mathbf{n}$ is the identity arrow at $\mathbf{n}$. Thus, W_φ is a category.

Since a pair $(m,n)\in\#\varphi$ may represent many arrows of W_φ, it is important at times to use a less ambiguous notation for arrows. In these cases, the arrow (1.1) will be denoted by $(n_\mathrm{L},(m,n),n'_\mathrm{R})$. Such notation completely specifies the initial object, $\mathbf{n}=(n_\mathrm{L},nn'_\mathrm{R})$, and the terminal object, $\mathbf{n}'=(n_\mathrm{L}n,n'_\mathrm{R})$ of the arrow. It should be noted that for any $n_1,n_2\in M\varphi$ and any $(m,n)\in\#\varphi$, there is an arrow $(n_1,(m,n),n_2)$ in W_φ. For example, the arrow $(m,n):(1,n)\to(n,1)$ is always present in W_φ when $(m,n)\in\#\varphi$.

We now define the congruence η on W_φ. This is done by associating to each arrow $(n_\mathrm{L},(m,n),n'_\mathrm{R}):\mathbf{n}\to\mathbf{n}'$ a function (of two variables)

$$(1.3)\qquad \begin{aligned}&[n_\mathrm{L},(m,n),n'_\mathrm{R}]:n_\mathrm{L}\varphi^{-1}\times n'_\mathrm{R}\varphi^{-1}\to M,\\ &m_\mathrm{L}[n_\mathrm{L},(m,n),n'_\mathrm{R}]m'_\mathrm{R}=m_\mathrm{L}mm'_\mathrm{R}.\end{aligned}$$

Here we are using the notation for functions of two variables discussed in the introduction. The element m_L belongs to $n_\mathrm{L}\varphi^{-1}$ and m'_R belongs to $n'_\mathrm{R}\varphi^{-1}$. Two coterminal arrows of W_φ are *equivalent* (mod η) if they define the same function (1.3).

We will show that η is a congruence by showing that it is both a right and a left congruence. To show that η is a right congruence, let $(m,n),(m',n'):\mathbf{n}\to\mathbf{n}'$ be coterminal arrows that are equivalent (mod η), and let $(m_0,n_0):\mathbf{n}'\to\mathbf{n}''$ be an arrow starting at object $\boldsymbol{n}'$. Showing that the composite arrows (mm_0, nn_0) and $(m'm_0, n'n_0)$ are equivalent (mod η) requires establishing the equation

$$(1.4) \qquad m_{\mathrm{L}} m m_0 m''_{\mathrm{R}} = m_{\mathrm{L}} m' m_0 m''_{\mathrm{R}}$$

for all $(m_{\mathrm{L}}, m''_{\mathrm{R}}) \in n_{\mathrm{L}}\varphi^{-1} \times n''_{\mathrm{R}}\varphi^{-1}$. Since $m_0 \in n_0\varphi^{-1}$ and $n_0 n''_{\mathrm{R}} = n'_{\mathrm{R}}$, we have

$$m_0 m''_{\mathrm{R}} \in n_0\varphi^{-1} \subseteq (n_0 n''_{\mathrm{R}})\varphi^{-1} = n'_{\mathrm{R}}\varphi^{-1}.$$

Equation (1.4) now follows from the assumption that the arrows (m,n) and $(m',n'): \boldsymbol{n} \to \boldsymbol{n}'$ are equivalent; that is, they define the same function (1.3). A dual argument is used to show that η is a left congruence.

In summary, the *kernel of* φ, denoted by K_φ, is the quotient category W_φ/η. The objects of K_φ are pairs of elements in the image of φ; that is,

$$\mathrm{Obj}(K_\varphi) = \boldsymbol{M\varphi}.$$

The arrows of K_φ are the functions (1.3). The hom-sets of K_φ are given by

$$K_\varphi(\boldsymbol{n}, \boldsymbol{n}') = \{[n_{\mathrm{L}}, (m,n), n'_{\mathrm{R}}] : n_{\mathrm{L}} n = n'_{\mathrm{L}} \text{ and } n_{\mathrm{R}} = n n'_{\mathrm{R}}\}.$$

Composition of consecutive arrows is given by the rule

$$[n_{\mathrm{L}}, (m,n), n'_{\mathrm{R}}][n'_{\mathrm{L}}, (m',n'), n''_{\mathrm{R}}] = [n_{\mathrm{L}}, (mm', nn'), n''_{\mathrm{R}}]$$

and the identity arrow at object $\boldsymbol{n}$ is the function $[n_{\mathrm{L}}, (1,1), n_{\mathrm{R}}]$.

Note that while the arrows of K_φ are described as functions (1.3), the composition in K_φ is not function composition. The functions (1.3) do not compose. This differs from the case of the derived category, D_φ, where the arrows are functions, and composition in D_φ is function composition. The derived category is the category of a concrete category; the kernel does not seem to have such a description.

Each object of K_φ, being a pair of elements in N, may be 'evaluated' by multiplication. That is, we may consider the evaluation function

$$\theta : \mathrm{Obj}(K_\varphi) \to N, \qquad \boldsymbol{n}\theta = n_{\mathrm{L}} n_{\mathrm{R}}.$$

It is interesting to note that this evaluation is constant along paths and connected components of K_φ. For, if there is an arrow $(m,n): \boldsymbol{n} \to \boldsymbol{n}'$ in W_φ, then by (1.1) we have

$$\boldsymbol{n}\theta = n_{\mathrm{L}} n_{\mathrm{R}} = n_{\mathrm{L}}(n n'_{\mathrm{R}}) = n'_{\mathrm{L}} n'_{\mathrm{R}} = \boldsymbol{n}'\theta.$$

Consequently, we may conclude that the category K_φ has as a least as many connected components as there are elements in $M\varphi$, the image of φ.

Alternate notation for the arrow $[n_{\mathrm{L}}, (m,n), n'_{\mathrm{R}}] \in K_\varphi(\boldsymbol{n}, \boldsymbol{n}')$ is

$$[m,n] : \boldsymbol{n} \to \boldsymbol{n}'$$

or, when ambiguity is not a problem, simply $[m,n]$. This arrow is said to be *represented by* $(m,n) \in \#\varphi$. In general, an arrow of K_φ may be represented by many elements of $\#\varphi$. However, our first lemma shows that arrows of the form

$[1,(m,n),1]:(1,n)\to(n,1)$, which always are present in K_φ, have unique representatives.

Lemma 1.1. *Let $\varphi: M\to N$ be a relation of monoids.*

(a) $[1,(m,n),1]=[1,(m',n'),1] \quad\Rightarrow\quad m=m'.$

(b) $K_\varphi((1,1))\approx 1\varphi^{-1}.$

Proof. (a) If $[1,(m,n),1]=[1,(m',n'),1]$, then by (1.3), $m_1mm_2=m_1m'm_2$ for all $m_1,m_2\in 1\varphi^{-1}$. Setting $m_1=m_2=1$ yields $m=m'$.

(b) $1\varphi^{-1}$ is clearly a submonoid of M, and the function

$$\theta: 1\varphi^{-1}\to K_\varphi((1,1)),\qquad m\theta=[1,(m,1),1]$$

is evidently a surjective morphism of monoids. In fact, because of (a), θ is an isomorphism. □

A relation $\varphi: X\to Y$ is *injective* if for all $x,x'\in X$, φ satisfies

$$x\varphi\cap x'\varphi\neq\emptyset \quad\Rightarrow\quad x=x'. \tag{1.5}$$

Equivalently, φ is an injective relation iff $\varphi^{-1}: Y\to X$ is a partial function. A relation $\varphi: X\to Y$ is *fully defined* if $x\varphi\neq\emptyset$ for all $x\in X$. For example, a division $\varphi: M\prec N$ of monoids is a relation of monoids that is both injective and fully defined on M. Equivalently, $\varphi: M\to N$ is a division iff $\varphi^{-1}: N\to M$ is a surjective partial function.

A *relational morphism* of categories $\varphi: S\lhd T$ is a relation of categories whose object relation is a function and whose hom-set relations are fully defined. A morphism (functor) of categories is an example of a relational morphism. A relational morphism φ is a *division of categories* if, further, each hom-set relation is injective. Note that category division, when restricted to monoids (categories with one object), coincides with monoid division. The reader should consult [8] for a full discussion of division of monoids and categories.

Division defines a preorder on categories, and by the usual techniques, induces an equivalence relation on categories. We write $S\sim T$ and say S is (divisionally) *equivalent* to T if both $S\prec T$ and $T\prec S$. This equivalence relation is *not* the same as the natural equivalence of category theory, but is more general. See [8] for more details. Equivalent finite monoids are isomorphic; such cannot be said of equivalent finite categories. In fact, it is quite common for a finite category to be equivalent to one of its subcategories. Trivial categories (posets) provide an example of this fact, below.

We call a category *trivial* if it has at most one arrow per hom-set, that is, if the category is a poset. If S is a trivial category, then the collapsing morphism $S\to\mathbf{1}$ is faithful, and thus is a division. On the other hand, $\mathbf{1}$ is a subcategory of S, and the inclusion morphism is a division. Thus S is trivial iff $S\sim\mathbf{1}$.

Proposition 1.2. *A relation φ of monoids is injective iff K_φ is trivial.*

Proof. Assume that $\varphi: M \to N$ is injective, and consider two coterminal arrows

$$[m,n],[m',n']: \mathbf{n} \to \mathbf{n}'$$

of K_φ. For any $m_L \in n_L\varphi^{-1}$, we have $m_L m \in (n_L n)\varphi^{-1}$ and $m_L m' \in (n_L n')\varphi^{-1}$. But $n_L n = n_L n' = n'_L$, so we have

$$n'_L \in (m_L m)\varphi \cap (m_L m')\varphi.$$

Since φ is injective, we deduce from (1.5) that $m_L m = m_L m'$ for all $m_L \in n_L\varphi^{-1}$. It follows from (1.3) that $[m,n]=[m',n']$. This argument shows that K_φ has at most one arrow per hom-set, that is, K_φ is a trivial category.

Conversely, assume that K_φ is trivial, and let $n \in m\varphi \cap m'\varphi$. The pair $(m,n) \in \#\varphi$ gives rise to the arrow

$$[1,(m,n),1]:(1,n) \to (n,1)$$

of K_φ. Similarly, $(m',n) \in \#\varphi$ defines $[1,(m',n),1]:(1,n) \to (n,1)$. Since K_φ is trivial, we deduce that $[1,(m,n),1]=[1,(m',n),1]$. Lemma 1.1 states that in this case $m=m'$. Therefore, by (1.5), φ is an injective relation. □

At the opposite extreme, we have the *collapsing* morphism $\varphi: M \to \mathbf{1}$. In this case K_φ has one object, $(1,1)$, so K_φ is the local monoid $K_\varphi((1,1))$. Since $M = 1\varphi^{-1}$, Lemma 1.1(b) proves

Proposition 1.3. *Let M be a monoid. If $\varphi: M \to \mathbf{1}$ is the collapsing morphism, then $K_\varphi \approx M$.* □

The inverse of a collapsing morphism is an injective relation. Propositions 1.2 and 1.3 combine to show that the kernel of a relation and the kernel of the inverse relation can differ by an arbitrary amount.

The two stages of the construction of the kernel, W_φ, then $\eta: W_\varphi \to K_\varphi$, are often useful when establishing a category division of the form $K_\varphi \prec V$. The pattern that occurs is this: First, a relational morphism $\psi: W_\varphi \lhd V$ is established. Then, since the inverse of the quotient morphism η is a relational morphism (actually a division), the composition $\eta^{-1}\psi: K_\varphi \to V$ defines a relational morphism. The second step is to show that $\eta^{-1}\psi$ is injective on each hom-set of K_φ. This establishes the division

$$\eta^{-1}\psi: K_\varphi \prec V.$$

The next topic, which shows how our kernel is related to the traditional group theory kernel, illustrates this procedure.

Proposition 1.4. *Let $\varphi: G \to H$ be the group morphism. Then $K_\varphi \sim \ker\varphi$.*

Proof. Using Lemma 1.1(b), we obtain $1\varphi^{-1} < K_\varphi$. For the opposite inquality, we will construct an faithful morphism of categories

$$\theta: W_\varphi \to 1\varphi^{-1}.$$

In particular, θ is a division. Since $\eta^{-1}: K_\varphi \to W_\varphi$ is also a division, the composition $\eta^{-1}\theta: K_\varphi < 1\varphi^{-1}$ provides the reverse inequality.

For each $h \in G\varphi$, choose an element $\bar{h} \in G$ with $\bar{h}\varphi = h$. Then for each hom-set $W_\varphi(h, h')$, define

$$(1.6) \qquad \theta: W_\varphi(h, h') \to 1\varphi^{-1}, \qquad (g, h)\theta = \bar{h}_{\mathrm{L}} g (\bar{h}'_{\mathrm{L}})^{-1}.$$

Because $h_{\mathrm{L}} h = h'_{\mathrm{L}}$, we have $(\bar{h}_{\mathrm{L}} g (\bar{h}'_{\mathrm{L}})^{-1})\varphi = h_{\mathrm{L}} h (h'_{\mathrm{L}})^{-1} = 1$. Therefore, (1.6) is well defined. If $(g, h)\theta = (g', h')\theta$, then by cancellation, $g = g'$. Since $h_{\mathrm{L}} h = h'_{\mathrm{L}} = h_{\mathrm{L}} h'$, we also have $h = h'$. This shows that (1.6) is an injective function. An easy argument now shows $\theta: W_\varphi \to \ker\varphi$ to be an injective morphism of categories. □

In the case of group morphisms, K_φ is actually a groupoid. That is, every arrow of K_φ is invertible. The inverse of $[h_1, (g, h), h_2]$ is $[h_1 h, (g^{-1}, h^{-1}), h h_2]$. However, with the exception of the kernels of collapsing morphisms, K_φ cannot be a connected groupoid. This is because, by earlier remarks, K_φ has at least card $H\varphi$ connected components. Consider, for example, the identity function on $\mathbb{Z}_2 = \{1, 0\}$. The connected components of the kernel are $\{(0,0), (1,1)\}$ and $\{(0,1), (1,0)\}$. In particular, this means that K_φ is not 'naturally equivalent' to $\ker\varphi$. Clearly however, K_φ is a coproduct of connected groupoids that are naturally equivalent to $\ker\varphi$.

The derived category, D_φ, of a relation of monoids $\varphi: M \to N$ was introduced in [8]. That construction provides the link between relations of monoids and the wreath product. We next show that the kernel is smaller than the derived category. To review, the objects of D_φ are the members of $M\varphi$, and the arrows of $D_\varphi(n_1, n_2)$ are functions of the form

$$[n_1, (m, n)]: n_1\varphi^{-1} \to n_2\varphi^{-1}, \qquad n_1 n = n_2, \qquad m_1[n_1, (m, n)] = m_1 m.$$

Composition is given by $[n_1, (m, n)][n_1 n, (m', n')] = [n_1, (mm', nn')]$, and the identity arrow at object n is $[n, (1, 1)]$. See [8] for a complete discussion.

Proposition 1.5. *Let $\varphi: M \to N$ be a relation of monoids. Then*

$$K_\varphi < D_\varphi.$$

Proof. We first establish a morphism $\theta: W_\varphi \to D_\varphi$ of categories. Define an object function

$$\theta: \mathrm{Obj}(W_\varphi) \to \mathrm{Obj}(D_\varphi), \qquad (n_{\mathrm{L}}, n_{\mathrm{R}})\theta = n_{\mathrm{L}}$$

and hom-set functions

$$\theta: W_\varphi(n, n') \to D_\varphi(n_{\mathrm{L}}, n'_{\mathrm{L}}), \qquad (m, n)\theta = [n_{\mathrm{L}}, (m, n)].$$

θ is easily seen to be a morphism.

Composing η^{-1} with θ, we obtain the relational morphism

$$\eta^{-1}\theta : K_\varphi \lhd D_\varphi .$$

Showing that $\eta^{-1}\theta$ is injective on hom-sets will establish the assertion.

Let $(m,n),(m',n')\in W_\varphi(n,n')$, and suppose that $(m,n)\theta=(m',n')\theta$. We must show that $(m,n)\eta=(m',n')\eta$, i.e., $[n_{\rm L},(m,n),n'_{\rm R}]=[n_{\rm L},(m',n'),n'_{\rm R}]$. Let $m_{\rm L}\in n_{\rm L}\varphi^{-1}$ and $m'_{\rm R}\in n'_{\rm R}\varphi^{-1}$. Since $[n_{\rm L},(m,n)]=[n_{\rm L},(m',n')]$, we have $m_{\rm L}m=m_{\rm L}m'$. Therefore, it follows that

$$m_{\rm L}mm'_{\rm R}=m_{\rm L}m'm'_{\rm R}.$$

The assertion follows from (1.3). □

The kernel and the reversal operation enjoy a pleasant relationship. Given a relation of categories $\varphi : S \to T$, the reverse relation $\varphi^\varrho : S^\varrho \to T^\varrho$ is defined by

$$c\varphi^\varrho = c\varphi \qquad \text{on objects,}$$

$$s^\varrho\varphi^\varrho=(s\varphi)^\varrho \quad \text{on arrows.}$$

Proposition 1.6. *Let $\varphi : M \to N$ be a relation of monoids. Then*

$$K_{\varphi^\varrho}\approx (K_\varphi)^\varrho.$$

Proof. The object function

$$\theta : \mathrm{Obj}(K_\varphi)\to \mathrm{Obj}(K_{\varphi^\varrho}), \qquad (n_{\rm L},n_{\rm R})\theta=(n_{{\rm R}^\varrho},n_{{\rm L}^\varrho})$$

and the hom-set functions

$$\theta : K_\varphi(n,n')\to K_{\varphi^\varrho}(n'\theta,n\theta), \qquad [n_{\rm L},(m,n),n'_{\rm R}]\theta=[n'_{{\rm R}^\varrho},(m^\varrho,n^\varrho),n_{{\rm L}^\varrho}]$$

combine to define a contravariant functor $\theta : K_\varphi \to K_{\varphi^\varrho}$. There results a covariant functor (i.e., a morphism)

$$\bar\theta : (K_\varphi)^\varrho \to K_{\varphi^\varrho}$$

which, in fact, is an isomorphism. □

The remaining results of this section are stated without proofs, since their proofs are straight-forward. The first concerns the direct product.

Proposition 1.7. *Let $\varphi_i : M_i \to N_i$, $i=1,2$, be the relations of monoids with corresponding kernels K_i. Then*

$$K_1\times K_2\approx K_\varphi$$

where φ is the product relation

$$\varphi_1 \times \varphi_2 : M_1 \times M_2 \to N_1 \times N_2,$$

$$(m_1, m_2)\varphi_1 \times \varphi_2 = \{(n_1, n_2) : n_i \in m_i \varphi_i,\ i = 1, 2\}.$$

A similar result holds for arbitrary large products of relations. □

An *admissible factorization* (α, W, β) of a relation $\varphi : M \to N$ of monoids is any factorization

$$\varphi = \alpha^{-1}\beta, \qquad \alpha : W \to M, \qquad \beta : W \to N$$

where W is a monoid, and α and β are morphisms. An admissible factorization of note is the *canonical factorization* $(\varphi_M, \#\varphi, \varphi_N)$, where φ_M and φ_N are the restrictions of the projection morphisms $\varrho_M : M \times N \to M$ and $\varrho_N : M \times N \to N$.

Proposition 1.8. *Let (α, W, β) be an admissible factorization for $\varphi : M \to N$. Then K_φ is a quotient of K_β. If $(\alpha, \#\varphi, \beta)$ is the canonical factorization for φ, then $K_\varphi \approx K_\beta$.* □

2. Finite simple monoids and P-free relations

The scope of the discussions in the next three sections will be limited to *finite* monoids and *finite* categories. These sections assume certain standard facts about finite monoids and semigroups. The reader is directed to Appendix A for a discussion of these facts.

A monoid $M \neq \mathbf{1}$ is *simple* if M has no non-trivial congruences. Equivalently, M is simple if its only morphic images are M and $\mathbf{1}$. The monoid $U_1 = \{1, 0\}$ is simple, and simple groups are, of course, simple monoids.

Proposition 2.1. *The finite simple monoids consist of U_1 and the finite simple groups.*

Proof. Let M be a simple finite monoid that is not a group. Let G be the maximal subgroup of M, and let I be the complement of G. Since M is finite, I is an ideal. Since M is simple, the quotient morphism $M \to M/I$ must be the identity; that is, I is a singleton. Thus we may write $M = G \cup \{0\}$. Now we may define the morphism

$$M \to U_1, \qquad g \to 1, \quad 0 \to 0.$$

Since this morphism must be an isomorphism, we have $M \approx U_1$. □

Semigroups cannot be avoided when dealing with monoids. An ideal of a monoid M is generally a subsemigroup of, not a monoid in, M. When $\varphi : M \to N$ is a relation of monoids and $e \in N$ is an idempotent, then $e\varphi^{-1}$ is generally a subsemigroup of M which is not a monoid. The latter situation causes us to expand our discussion to include semigroups at this point. The interplay between monoid division and semigroup division is discussed in [8].

Simple monoids have the following important property:

Proposition 2.2. *Let $\varphi : S \to T$ be a morphism of semigroups, and let $P < S$, where P is a simple monoid. Then either $P < T$ or $P < e\varphi^{-1}$ for some idempotent $e \in T$.*

Proof. Suppose $U_1 < S$. Since S is finite, S has a copy of U_1 as a subsemigroup. Because U_1 is simple, either $U_1\varphi \approx U_1$, in which case $U_1 < T$, or $U_1\varphi$ is a singleton idempotent e, in which case $U_1 < e\varphi^{-1}$.

Now let P be a simple group and suppose $P < S$. Since S is finite, there is a group G in S and a surjective morphism $\theta : G \to P$. Let e be the identity of the group $G\varphi$ in T, and let $N = e\varphi^{-1} \cap G$. Then N is a normal subgroup of G, $G/N \approx G\varphi < T$ and $N < e\varphi^{-1}$.

Let $N' = N\theta$. Then N' is a normal subgroup of P, and P/N' is a quotient of G/N. In terms of division we have $P/N' < G/N$ and $N' < N$. Therefore, $P/N' < T$ and $N' < e\varphi^{-1}$. Now, since P is simple, either $P/N' \approx P$, in which case $P \approx P/N' < T$, or $N' = P$, in which case $P = N' < e\varphi^{-1}$. □

Corollary 2.3. *Let P be a simple monoid, and let S and T be semigroups. Then*

$$P < S \times T \quad \Rightarrow \quad P < S \text{ or } P < T.$$

Proof. Assume that P divides $S \times T$, but that P does not divide T. Let $\pi : S \times T \to T$ be the projection. Then by Proposition 2.2, $P < e\pi^{-1}$ for some idempotent $e \in T$. Furthermore, the map

$$e\pi^{-1} \to S, \qquad (s, e) \to s$$

is an injection. Therefore, $P < S$. □

Let P be a simple monoid. A semigroup S is called *P-free* if P does not divide S. More generally, let $\mathscr{P}$ be a collection of simple monoids. Then a semigroup S is *$\mathscr{P}$-free* if no member of $\mathscr{P}$ divides S, that is, S is P-free for each $P \in \mathscr{P}$.

It should be noted that for the sake of this definition, the family $\mathscr{P}$ may as well be closed under domination. That is, if $\mathscr{P}$ is any collection of simple monoids and $\mathscr{P}' = \{Q \text{ simple: } P < Q \text{ for some } P \in \mathscr{P}\}$, then a semigroup is $\mathscr{P}$-free iff it is $\mathscr{P}'$-free.

Every semigroup is $\mathscr{P}$-free when $\mathscr{P} = \emptyset$. On the other hand, when $\mathscr{P}$ = all simple monoids, then the $\mathscr{P}$-free semigroups are those semigroups whose monoids are trivial. This class is usually denoted by **L1**. If $\mathscr{P}$ consists of all simple groups, then the $\mathscr{P}$-free semigroups are the aperiodic semigroups, that is, the semigroups with only trivial group divisors.

For any $\mathscr{P}$, the collection of all $\mathscr{P}$-free semigroups forms an S-variety. This collection is clearly closed under division, and Corollary 2.3 shows that $\mathscr{P}$-free semigroups admit finite direct products.

$\mathscr{P}$-free semigroups lead to useful classifications of monoid relations. Let P be a simple monoid. A relation $\varphi: M \to N$ of monoids is *P-free* if each of the subsemigroups $\{e\varphi^{-1}: e^2 = e \in N\}$ of M is P-free. More generally, if $\mathscr{P}$ is a collection of simple monoids, then a relation $\varphi: M \to N$ is $\mathscr{P}$-free if φ is P-free for each $P \in \mathscr{P}$.

Every relation is $\mathscr{P}$-free when $\mathscr{P} = \emptyset$. If $\mathscr{P}$ consists of all simple groups, then the $\mathscr{P}$-free relations are the aperiodic relations of monoids. When $\mathscr{P}$ = all simple monoids, the $\mathscr{P}$-free relations are relations that are both aperiodic and U_1-free. We now establish important properties of $\mathscr{P}$-free relations.

Proposition 2.4. *Let $(\alpha, \#\varphi, \beta)$ be the canonical factorization of a relation $\varphi: M \to N$. Then φ is $\mathscr{P}$-free iff β is $\mathscr{P}$-free.*

Proof. Let $e \in N$ be an idempotent. Since $\varphi = \alpha^{-1}\beta$, we have $e\varphi^{-1} = (e\beta^{-1})\alpha$. However, $e\beta^{-1} = \{(m, e) \in \#\varphi\}$, and α is the projection $\pi: M \times N \to M$ restricted to $\#\varphi$. Consequently, $\alpha: e\beta^{-1} \to e\varphi^{-1}$ is an isomorphism, and $e\varphi^{-1}$ is $\mathscr{P}$-free iff $e\beta^{-1}$ is $\mathscr{P}$-free. □

Proposition 2.5. *Let $\varphi_1: M \to M'$ and $\varphi_2: M' \to M''$ be surjective morphisms. If the composition $\varphi_1\varphi_2$ is $\mathscr{P}$-free, then both φ_1 and φ_2 are $\mathscr{P}$-free.*

Proof. Let $\varphi = \varphi_1\varphi_2$, and let e be an idempotent in M''. Then $e\varphi^{-1}$ is $\mathscr{P}$-free. Since $(e\varphi^{-1})\varphi_1 = e\varphi_2^{-1}$, we see that $e\varphi_2^{-1} < e\varphi^{-1}$. Therefore, $e\varphi_2^{-1}$ is $\mathscr{P}$-free, and consequently, φ_2 is $\mathscr{P}$-free.

Let $f \in M'$ be an idempotent. Then $f\varphi_2 = e$ is an idempotent in M'', and $f\varphi_1^{-1} \subseteq e\varphi^{-1}$. Since $e\varphi^{-1}$ is $\mathscr{P}$-free, it follows that φ_1 is $\mathscr{P}$-free. □

Proposition 2.6. *Let $\varphi: M \to N$ be a relation of monoids. Then φ is $\mathscr{P}$-free iff for each $\mathscr{P}$-free subsemigroup N' of N, the subsemigroup $N'\varphi^{-1}$ is $\mathscr{P}$-free.*

Proof. Assume φ is $\mathscr{P}$-free, and let N' be a $\mathscr{P}$-free subsemigroup of N. Let $(\alpha, \#\varphi, \beta)$ be the canonical factorization of φ. Then by Proposition 2.4, β is $\mathscr{P}$-free. Suppose $P < N'\beta^{-1}$ for some simple monoid P. Applying Proposition 2.2 to $\beta: N'\beta^{-1} \to N'$, we see that either $P < N'$ or $P < e\beta^{-1}$ for some idempotent $e \in N'$. Since both N' and β are $\mathscr{P}$-free, P cannot belong to $\mathscr{P}$. Therefore, $N'\beta^{-1}$ is $\mathscr{P}$-free. Since $N'\varphi^{-1} = (N'\beta^{-1})\alpha$, it follows that $N'\varphi^{-1}$ is $\mathscr{P}$-free. The converse is immediate. □

Let $\varphi: M \to M'$ and $\psi: M' \to M''$ be $\mathscr{P}$-free relations, and let $e \in M''$ be an idempotent. Then $e\psi^{-1}$ is $\mathscr{P}$-free and $e(\varphi\psi)^{-1} = (e\psi^{-1})\varphi^{-1}$. Thus, by Proposition 2.6, $e(\varphi\psi)^{-1}$ is $\mathscr{P}$-free. This proves

Proposition 2.7. *$\mathscr{P}$-free relations are closed under composition.* □

We now introduce two classes of relations which we will consider 'primitive'.

These classes will be shown, in the next section, to generate all relations of monoids. We first present a useful lemma.

Lemma 2.8. *Let $\varphi: M \to N$ be a relation of monoids, and suppose a monoid $P < e\varphi^{-1}$ for some idempotent $e \in N$. Then P divides a local monoid of K_φ.*

Proof. Since $P < e\varphi^{-1}$, there is a monoid $M' \subseteq e\varphi^{-1}$ that P divides. Define the function

$$\theta: M' \to K_\varphi(e,e), \qquad m\theta = [e,(m,e),e].$$

If $m\theta = m'\theta$, then by (1.3), $amb = am'b$ for all $a, b \in e\varphi^{-1}$. In particular, if f is the identity of M', then $m = fmf = fm'f = m'$. Therefore, θ is injective. Since $m\theta m'\theta$ clearly equals $(mm')\theta$, we see that θ is an injective morphism of semigroups. Therefore, $P < M' < K_\varphi(e,e)$. □

The first type of primitive relation is now introduced. A relation $\varphi: M \to N$ will be called *locally trivial* if its kernel K_φ is a locally trivial category, that is, a category with trivial local monoids. The collection of all locally trivial categories is denoted $\ell\mathbf{1}$, and forms a $\mathbf{C}$-variety.

Recall that a $\mathbf{C}$-variety is a collection of finite categories that admit division and finite direct products. If K is a category, then (K) denotes the $\mathbf{C}$-variety generated by K. The smallest $\mathbf{C}$-variety, $(\mathbf{1})$, is the collection of all trivial categories, that is, those categories equivalent to $\mathbf{1}$. Proposition 1.2 showed that a relation has a trivial kernel iff the relation is injective.

The next smallest $\mathbf{C}$-variety is $\ell\mathbf{1}$. That is, if $\mathbf{W}$ is a $\mathbf{C}$-variety, then either

$$\mathbf{W} = \{\mathbf{1}\} \quad \text{or} \quad \ell\mathbf{1} \subseteq \mathbf{W}.$$

This fact is deep and is established in [8, Theorem 8.1]. Consequently, if K is a locally trivial category that is not trivial, then $(K) = \ell\mathbf{1}$.

In summary, if a relation φ is locally trivial, then either φ is an injective relation, or $(K_\varphi) = \ell\mathbf{1}$.

The second type of primitive relation is now introduced. Let Q be a simple monoid. A relation $\varphi: M \to N$ of monoids is *primitive of type Q* if

(2.1) $\qquad \varphi$ is not Q-free, and

(2.2) $\qquad K_\varphi \in (Q)$.

If φ is primitive of type Q, then by (2.1), $Q < e\varphi^{-1}$ for some idempotent $e \in N$. It follows from Lemma 2.8 that $Q < K_\varphi$. Combining this inequality with (2.2) gives us the following fact:

(2.3) $\qquad$ If φ is primitive of type Q, then $(K_\varphi) = (Q)$.

However, the converse is false; an example is provided in Example 2.11.

In order to establish (2.2), it suffices to show that each local monoid of K_φ

belongs to (Q). In other words, we may replace (2.2) with

(2.2′) Local monoids of K_φ belong to (Q).

This equivalence is by no means automatic. It is based on results from [8] which may be summarized as follows:

Theorem 2.9. *Let Q be a simple monoid and let K be a category. Then*

$$K \in (Q) \quad \textit{iff} \quad \textit{each local monoid of } K \textit{ belongs to } (Q). \qquad \square$$

The $Q = U_1$ case is deep and is based on work of Simon. The case when Q is a group is based on results of Thérien. The original formulations of these ideas did not use the language of categories and category division. For the original references and the category formulations of these theorems, see [8, Example 15.6] for the U_1 case, and [8, Propositions 11.6 and 13.8] for the group case.

A relation $\varphi : M \to N$ of monoids will be called *primitive* if either φ is locally trivial or φ is primitive of type Q, where Q is any simple monoid. The relationship between primitive relations and the notion of P-free is given in the next proposition.

Proposition 2.10. *Let $\varphi : M \to N$ be a relation of monoids.*

(a) *If φ is locally trivial, then φ is P-free for all simple monoids P.*

(b) *If φ is primitive of type Q and P is a simple monoid, then φ is P-free iff P does not divide Q.*

Proof. (a) If φ is locally trivial, the local monoids of K_φ are trivial. It follows from Lemma 2.8 that for each idempotent $e \in N$, the monoid divisors of $e\varphi^{-1}$ are trivial. Therefore, φ is P-free for all simple monoids P.

(b) Let φ be primitive of type Q, and let P be a simple monoid. Then by (2.1), $Q < e\varphi^{-1}$ for some idempotent $e \in N$. Therefore, if $P < Q$, φ can not be P-free. Conversely, if φ is not P-free, then $P < e\varphi^{-1}$ for some idempotent $e \in N$. It follows from Lemma 2.8 that P divides K_φ. Then by (2.2) we have

$$P < Q \times \cdots \times Q \quad \text{(finite)}.$$

Since P is a simple monoid, Corollary 2.3 implies that $P < Q$. $\square$

Example 2.11. This example supports the remark following (2.3). Let M be the cyclic monoid defined by $x^4 = x^3$. Thus $M = \{1, x, x^2, x^3\}$. Consider the morphism $\theta : M \to U_1$ defined by $x\theta = 0$. Then $1\theta^{-1} = \{1\}$ and $0\theta^{-1} = \{x, x^2, x^3\}$, so θ is P-free for every simple monoid P. In particular, this means that θ is not primitive of type U_1. We show that $(K_\theta) = (U_1)$.

We calculate each local monoid of K_θ. By (1.1), an arrow $(m, n) \in W_\theta(n_L, n_R)$ must satisfy $n_L n = n_L$ and $n n_R = n_R$. If either n_L or n_R equals 1, then $(1, 1)$ is the only member of $W_\theta(n_L, n_R)$. It follows that each local monoid of K_θ is trivial except possibly $K_\theta(0,0)$. In this last case, every member of $\#\theta$ defines an arrow in $W_\theta(0,0)$. To determine $K_\theta(0,0)$, we must consider the function $[m,n]:0\theta^{-1}\times 0\theta^{-1}$ for each $(m,n)\in\#\theta$ as defined by (1.3). The function $[1,1]$ maps (x^i,x^j) to x^{i+j}, $i,j=1,2,3$. On the other hand, for each $k=1,2,3$, $[x^k,0]$ is the constant function with value x^3. It follows that $K_\theta(0,0)\approx U_1$.

From the above calculations, we see that (i) every local monoid of K_θ belongs to (U_1), and (ii) $U_1 \prec K_\theta$. Using Theorem 2.9, we conclude that $(K_\theta)=(U_1)$, even though θ is not primitive of type U_1.

The morphism $\theta:M\to U_1$ is P-free for all simple monoids P, yet θ is not locally trivial. We show that θ can be factored into locally trivial morphisms. Let $N=\{1,y,y^2\}$ with $y^3=y^2$. Then θ can be written $\theta=\theta_1\theta_2$, where $\theta_1:M\to N$ is given by $x\theta_1=y$, and where $\theta_2:N\to U_1$ is given by $y\theta_2=0$. An easy calculation shows that both θ_1 and θ_2 are locally trivial. This is a glimpse of Theorem 3.1, upcoming.

3. Prime decomposition of finite relations

We now present our main result about relations. All monoids and categories in this section are assumed finite.

Let $\varphi:M\to N$ be a relation of monoids. A *decomposition* of φ is a representation of φ as a composition of relations

$$\varphi=\varphi_1\cdots\varphi_k,\quad k\ge 1$$

where (i) the *factors* $\varphi_i:M_{i-1}\to M_i$ are relations of monoids with $M_0=M$ and $M_k=N$, (ii) each factor, except for possibly φ_1, is fully defined, and (iii) each factor, except for possibly φ_k, is surjective. In other words, the domain of each factor coincides with the range of the previous factor in the decomposition. Of course, any composition of the form $\varphi=\varphi_1\cdots\varphi_k$ may, by restriction, be refined term by term to produce a decomposition of φ.

For any collection $\mathscr{P}$ of simple monoids, we may consider the collection of all $\mathscr{P}$-free relations. Our result states that every relation in the collection has a decomposition within the collection with primitive factors.

Theorem 3.1. *Let $\varphi:M\to N$ be a relation of monoids, and let $\mathscr{P}$ be a collection of simple monoids. Then φ is $\mathscr{P}$-free iff φ has a decomposition whose factors are primitive $\mathscr{P}$-free relations.*

Since $\mathscr{P}$-free relations are closed under composition (Proposition 2.7), one direction of Theorem 3.1 is immediate. This section and the next are devoted to the proof of the other direction. We first present some important corollaries.

Every relation is $\mathscr{P}$-free when $\mathscr{P}=\emptyset$. Applying Theorem 3.1 to this case gives us the first corollary.

Corollary 3.2. *Every relation of monoids has a decomposition whose factors are primitive relations.* □

According to Proposition 2.10, a primitive relation that is U_1-free must either be locally trivial or must be primitive of type Q, where U_1 does not divide Q. Since U_1 divides no group, we obtain the following result from Theorem 3.1 when $\mathscr{P} = \{U_1\}$:

Corollary 3.3. *A relation $\varphi : M \to N$ is U_1-free iff φ has a decomposition whose factors are either locally trivial or primitive of type Q, Q a simple group.* □

On the other hand, no non-trivial group divides U_1. Thus, setting $\mathscr{P}$ to be all simple groups in Theorem 3.1 yields the following:

Corollary 3.4. *A relation $\varphi : M \to N$ is aperiodic iff φ has a decomposition whose factors are either locally trivial or primitive of type U_1.* □

Finally, no primitive relation of type Q is P-free for all simple monoids P. Letting $\mathscr{P}$ be all simple monoids gives us

Corollary 3.5. *A relation $\varphi : M \to N$ is aperiodic and U_1-free iff φ has a decomposition whose factors are locally trivial.* □

Another result of interest can be obtained from Corollaries 3.2, 3.3 and 3.4. Corollaries 3.3 and 3.4 show that every primitive relation is either aperiodic or U_1-free. The next proposition then follows from Corollary 3.2.

Proposition 3.6. *Every relation of monoids has a decomposition whose factors are either aperiodic or U_1-free relations.* □

We now begin our proof of the decomposition portion of Theorem 3.1. A morphism $\varphi : M \to N$ of monoids is *proper* if it is not injective. A morphism $\varphi : M \to N$ is an *MPS* (Maximal Proper Surmorphism) if (i) φ is proper, (ii) φ is surjective, and (iii) whenever φ has a decomposition $\varphi = \varphi_1\varphi_2$, where φ_1 and φ_2 are morphisms, then either φ_1 or φ_2 is not proper. Clearly, every proper surjective morphism has a decomposition where each factor is an MPS. The next proposition reduces the proof of Theorem 3.1 to the MPS case.

Proposition 3.7. (a) *If φ is an MPS that is U_1-free, then φ is a primitive morphism.*

(b) *If φ is an MPS that is not U_1-free, then φ has a decomposition $\varphi = \theta\psi$, where θ is a primitive relation of type U_1, and ψ is a locally trivial morphism.*

Proposition 3.7 is proved in the next section. We assume its truth for the time being, and complete the proof of Theorem 3.1.

Let $\varphi : M \to N$ be a $\mathscr{P}$-free relation of monoids, where $\mathscr{P}$ is a collection of simple monoids. We proceed to show that φ has a decomposition

$$\varphi = \gamma\beta_1 \cdots \beta_n\delta, \quad n \geq 0 \tag{3.1}$$

where γ and δ are injective relations and $\beta_1, \ldots, \beta_n$ are $\mathscr{P}$-free MPS's.

Let $(\alpha, \#\varphi, \beta)$ be the canonical factorization of φ. Then $\varphi = \alpha^{-1}\beta$ is a decomposition, and by Proposition 2.4, β is a $\mathscr{P}$-free morphism. Set $\gamma = \alpha^{-1}$; since α is a morphism, γ is an injective relation. If β is injective, then set $\delta = \beta$ and (3.1) is satisfied with $n = 0$. Otherwise, the morphism β has a decomposition $\beta = \beta'\delta$, where $\beta' : \#\varphi \to M\varphi$ is a proper surjective morphism, and $\delta : M\varphi \to N$ is the inclusion map. Clearly, α, β', and γ are $\mathscr{P}$-free.

Since β' is proper and surjective, it has a decomposition $\beta' = \beta_1 \cdots \beta_n$, where each factor β_i is an MPS. Furthermore by Proposition 2.5, each factor β_i must be $\mathscr{P}$-free. This establishes (3.1).

Now using (3.1), it suffices to establish the assertion of Theorem 3.1 for $\mathscr{P}$-free MPS's. Thus, let φ be a $\mathscr{P}$-free MPS. If φ is U_1-free, then according to Proposition 3.7, φ is already primitive. If φ is not U_1-free, then φ has a decomposition $\varphi = \theta\psi$, where θ is primitive of type U_1, and ψ is locally trivial. But since φ is not U_1-free, U_1 cannot belong to $\mathscr{P}$. Thus $\mathscr{P}$ consists of simple groups, and no member of $\mathscr{P}$ divides U_1. It follows from Proposition 2.10 that both θ and ψ are $\mathscr{P}$-free. Thus φ has a decomposition whose factors are primitive $\mathscr{P}$-free relations. □

Before proceeding with the proof of Proposition 3.7, we present an example that shows the necessity of relations in this theory. Note that Proposition 3.7 does *not* state that every MPS is primitive. If that were the case, then there would be no reason for developing this theory beyond the morphism (function) level. The word 'relation' could be replaced by 'morphism' everywhere in this section. And, in fact, this can be done for Corollaries 3.3 and 3.5. But it cannot be done in part (b) of Proposition 3.7, and thus, cannot be done in Theorem 3.1 and Corollaries 3.2 and 3.4. We present an MPS that is not primitive.

Example 3.8. Consider the monoid of all partial injective functions on two letters $\{x, y\}$, minus the permutation that interchanges the letters. This monoid, therefore, consists of the identity function, 1, and an ideal I consisting of four partial injective functions with singleton domains and the empty partial function, 0. This monoid is aperiodic and often goes by the name B_2. When the elements of I are identified, one obtains the quotient morphism

$$\varphi : B_2 \to U_1, \qquad 1\varphi = 1, \qquad I\varphi = 0. \tag{3.2}$$

Direct calculations reveal that φ is an MPS; no two elements of I can be identified by a congruence without all the elements of I being identified. Furthermore, since I contains copies of U_1, φ is not U_1-free.

We will show that $B_2 < K_\varphi$. Since B_2 is neither idempotent nor commutative, this will show that K_φ does not belong to (U_1). Nor is B_2 a group, so K_φ cannot belong to (Q) for any simple group Q. In short, φ cannot be primitive if $B_2 < K_\varphi$.

We establish $K_\varphi(0,0) \approx B_2$. Note first that every pair $(m,n) \in \# \varphi$ defines an arrow $(0,(m,n),0)$ of $W_\varphi(0,0)$. That is, $W_\varphi(0,0) \approx B_2$. Second, by the above remarks, the quotient $\eta : W_\varphi \to K_\varphi$ when restricted to $B_2 \approx W_\varphi(0,0)$, is either an isomorphism or must identify all the elements of I. We show the latter cannot happen. Let $a, b : \{x, y\} \to \{x, y\}$ be given by

$$xa = x, \qquad xb = y,$$

$$ya = \emptyset, \qquad yb = \emptyset.$$

According to (1.3), $(a,0)\eta = (b,0)\eta$ iff $mam' = mbm'$ for all $m \in I = 0\varphi^{-1}$. However, letting $m = m' = a$, we see that $aaa = a$ while $aba = 0$. This shows that η restricted to $W_\varphi(0,0)$ is an isomorphism.

Proposition 3.7(b) states that the MPS defined in (3.2) can be further decomposed into a relation θ followed by a morphism ψ, where θ is primitive of type U_1 and ψ is locally trivial. Since φ is an MPS, θ cannot be a morphism.

4. The proof of Proposition 3.7

Proposition 3.7 reduced the proof of Theorem 3.1 to the MPS case. In order to proceed, we will need facts about MPS's that appear in [5]. In order to state these facts, we first introduce some notation.

Let $\varphi : M \to N$ be a relation of monoids, and let X and Y be subsets of M. We say that φ *separates* X and Y if $X\varphi \cap Y\varphi = \emptyset$. Note that φ is injective on a subset X of M iff φ separates every pair of elements of X.

Let J be a $\mathcal{J}$-class of a monoid M. J partitions M into three disjoint subsets, $A(J)$, J, and $B(J)$, as follows:

$$A(J) = \{m \in M : m > J\}, \qquad B(J) = M - A(J) - J.$$

$A(J)$ is the set of elements of M that are strictly above J in the $\mathcal{J}$-ordering of M. $B(J)$ is the set of elements that are not above J and not in J. Note that both $J \cup B(J)$ and $B(J)$ are ideals of M.

A relation $\varphi : M \to N$ of monoids is *$\mathcal{J}$-singular* if there exists a $\mathcal{J}$-class J of M with the properties (i) φ is injective on $M - J$ and (ii) φ separates J and $A(J)$. If these conditions are met, then J is called a *singular* $\mathcal{J}$-class of φ.

The following facts are proved in [5]:

(4.1) Every MPS is $\mathcal{J}$-singular.

(4.2) Every MPS $\varphi : M \to N$ is either injective on the $\mathcal{H}$-classes of M or separates the $\mathcal{H}$-classes of M.

A third fact about MPS's is needed here. Before presenting this fact, we state some results from the classical structure theory of regular $\mathcal{J}$-classes of finite monoids. See Appendix A, [7] or [1] for more details.

Let M be a finite monoid, and let J be a regular $\mathcal{J}$-class of M. Then each of the $\mathcal{H}$-classes $\{H_1, \ldots, H_n\}$ of M in J are in 1:1 correspondence. In fact, let G be a maximal group of M in J with identity e. G is one of the $\mathcal{H}$-classes of J. Then the 1:1 correspondence can be stated as follows. For each $i \in \{1, \ldots, n\}$, there exist elements $a_i, b_i, c_i, d_i \in M$ satisfying

$$(4.3)\qquad \begin{aligned} &a_i H_i b_i = G, \qquad c_i G d_i = H_i, \qquad d_i b_i = e = a_i c_i, \\ &h b_i d_i = h = c_i a_i h \quad \forall h \in H_i. \end{aligned}$$

Furthermore, since e is the identity of G, these elements may be chosen so that $a_i, d_i \in eM \cap J$ and $b_i, c_i \in Me \cap J$. These facts are utilized in upcoming propositions.

Lemma 4.1. *Let $\varphi : M \to N$ be an MPS. Then φ restricted to any maximal group in M is either injective or is an MPS of groups.*

Proof. Suppose G is a maximal group in M and φ is not injective on G. Let J be the $\mathcal{J}$-class of M containing G; then J is regular and we may utilize (4.3). Furthermore, since φ is not injective on J, it follows that J must be a singular $\mathcal{J}$-class of φ.

Let $\sim$ be a congruence on G that refines φ; that is, $g \sim g'$ implies $g\varphi = g'\varphi$. Extend $\sim$ to M by the rule

$$(4.4)\qquad \begin{aligned} m \sim m' \quad \text{iff} \quad & m = m' \text{ or, for some } i \in \{1, \ldots, n\}, \\ & m, m' \in H_i \text{ and } a_i m b_i \sim a_i m' b_i \text{ (in } G). \end{aligned}$$

We will show that $\sim$ is a congruence on M that refines φ. Since φ is an MPS, this will mean that either $\sim$ is the identity congruence or $\sim$ coincides with φ. In particular, this will show that φ restricted to G cannot be refined non-trivially; that is, φ restricted to G is an MPS of groups.

The relation $\sim$ is certainly an equivalence relation. We first show that $\sim$ refines φ. Let $m \sim m'$ with $m \neq m'$. Then $m, m' \in H_i$ for some i, and $a_i m b_i \sim a_i m' b_i$. Let $g = a_i m b_i$ and $g' = a_i m' b_i$. Then since $g \sim g'$ in G, we have $g\varphi = g'\varphi$. But by (4.3), we have

$$m = c_i g d_i \quad \text{and} \quad m' = c_i g' d_i$$

so it follows that $m\varphi = m'\varphi$. Thus $\sim$ refines φ.

Last, to show that $\sim$ is a congruence on M, let $m \sim m'$ with $m, m' H_i$ for some i, and let $x, y \in M$. We must show that $xmy \sim xm'y$. Since $m \sim m'$, we have $m\varphi = m'\varphi$, and hence

$$(4.5)\qquad (xmy)\varphi = (xm'y)\varphi.$$

Since φ is not injective on G, and since G is an $\mathcal{H}$-class of M, it follows from (4.2) that φ separates the $\mathcal{H}$-classes of M. Furthermore, φ is injective on $M - J$. From (4.5) we must conclude that either $xmy = xm'y$, in which case we are done, or xmy, $xm'y \in H_j$ for some j. Assume the latter. According to (4.4) we need to establish

(4.6) $\qquad a_j xmyb_j \sim a_j xm'yb_j \quad (\text{in } G).$

Since $m, m' \in H_i$, we may write

(4.7) $\qquad a_j xmyb_j = ha_i mb_i h', \qquad a_j xm'yb_j = ha_i m'b_i h'$

where $h = a_j xc_i$ and $h' = d_i yb_j$. It follows from (4.3) that h and h' belong to $eMe \subseteq G \cup B(J)$. However, because of (4.7), neither h nor h' belong to $B(J)$. Thus, $h, h' \in G$ and (4.6) follows from (4.7). This shows that $\sim$ is a congruence and establishes the assertion. □

Let $\omega : G \to H$ be an MPS of groups. Then $\ker \omega$ is a minimal normal subgroup of G, and by a well-known fact of group theory, must be a direct product of a Jordan–Hölder factor, Q, of G. It follows from Proposition 1.4 that ω is primitive of type Q. We extend this result to MPS's that are not aperiodic.

Proposition 4.2. *Let $\varphi : M \to N$ be an MPS that is not aperiodic. Then φ is primitive of type Q, where Q is a Jordan–Holder factor of a maximal group in M.*

Proof. Since φ is not aperiodic, φ identifies some elements of a maximal group G of M. Let ω denote φ restricted to G. Then by Lemma 4.1, ω is an MPS of groups, and $\ker \omega \approx Q \times \cdots \times Q$ for some Jordan–Hölder factor of G. Since $Q < \ker \omega$, φ is not Q-free. In order to show that φ is primitive of type Q, we must show that K_φ belongs to (Q). Since $K_\varphi < D_\varphi$ (Proposition 1.5), and since $\ker \omega \approx Q \times \cdots \times Q$, it suffices to show that $D_\varphi < \ker \omega$. In light of Theorem 2.9, we need only show that every local monoid of D_φ divides $\ker \omega$.

Let $n_0 \in N$, and consider the monoid $W = \{(m, n) \in \#\varphi : n_0 n = n_0\}$. The local monoid $D_\varphi(n_0)$ is isomorphic to $W/{\equiv}$, where $\equiv$ is the congruence

$$(m, n) \equiv (m', n') \quad \text{iff} \quad m_0 m = m_0 m' \quad \forall m_0 \in n_0 \varphi^{-1}.$$

It follows directly that if $n_0 \varphi^{-1}$ is a singleton, then $D_\varphi(n_0)$ is trivial.

Let J be the $\mathcal{J}$-class of M containing G. Then J is regular and is a singular $\mathcal{J}$-class of φ. It follows that if n_0 does not belong to $J\varphi$, then $n_0 \varphi^{-1}$ is a singleton and $D_\varphi(n_0)$ is trivial. In particular, $D_\varphi(n_0) < \ker \omega$.

Thus, we may assume that $n_0 \in J\varphi$. Since φ is not injective on $\mathcal{H}$-classes of M, φ must separate $\mathcal{H}$-classes. Thus $n_0 \varphi^{-1}$ is contained in an $\mathcal{H}$-class of J. Applying the notation of (4.3) to J, we may conclude that $n_0 \varphi^{-1} \subseteq H_i$ for some $i \in \{1, \ldots, n\}$. Define the function

$$\theta : W \to \ker \omega, \qquad (m, n)\theta = d_i m' b_i.$$

To show that θ takes its values in $\ker \omega$, let $m_0 \in n_0 \varphi^{-1}$. Since $n_0 n = n_0$, it follows that $(m_0 m)\varphi = m_0 \varphi = n_0$ and $m_0 m \in H_i$. We may write, using (4.3),

(4.8) $\qquad a_i m_0 m b_i = (a_i m_0 b_i)(d_i m b_i).$

From (4.3) we know that $d_i mb_i \in eMe \subseteq G \cup B(J)$, and (4.8) shows that only $d_i mb_i \in G$ is possible. Furthermore, $a_i m_0 b_i$ and $a_i m_0\, mb_i \in G$, so we may apply φ to (4.8) and conclude that $d_i mb_i \in \ker \omega$.

In order to show that θ is a morphism, we need to establish the equation $d_i mb_i d_i m'b_i = d_i mm'b_i$. This will be done by proving that $d_i mb_i d_i = d_i m$. Let $m_0 \in n_0\varphi^{-1}$. Since $m_0 \in H_i$, (4.3) shows that $d_i \geq m_0$ in the $\mathscr{L}$-ordering of M. But since $d_i \mathscr{J} m_0$, it follows that $d_i \mathscr{L} m_0$. Choose $x \in M$ so that $d_i = xm_0$. Then

$$\begin{aligned} d_i mb_i d_i &= xm_0 mb_i d_i \\ &= xm_0 m \quad (\text{because } m_0 m \in H_i \text{ (4.3)}) \\ &= d_i m. \end{aligned}$$

Therefore, $\theta : W \to \ker \omega$ is a morphism.

To establish $D_\varphi(n_0) \prec \ker \omega$, it suffices to show that if $(m,n), (m',n') \in W$ with $(m,n)\theta = (m',n')\theta$, then $(m,n) \equiv (m',n')$. Let $m_0 \in n_0\varphi^{-1}$. Then $m_0 m \in n_0\varphi^{-1}$, so by (4.3) we may write

$$m_0 m = m_0 b_i d_i mb_i d_i = m_0 b_i (m,n)\theta d_i.$$

Since $(m,n)\theta = (m',n')\theta$, it follows that $m_0 m = m_0 m'$. Therefore, $D_\varphi(n_0) \prec \ker \omega$, and the assertion is established. □

The following lemma is needed for our next result:

Lemma 4.3. *Let $\varphi : M \to N$ be a morphism of monoids, and let J be a regular $\mathscr{J}$-class of M.*

(a) *If φ is aperiodic, then φ is injective on the $\mathscr{H}$-classes of J.*

(b) *If φ is U_1-free, then φ separates J and $B(J)$.*

Proof. (a) Let G be a maximal group in J. Since φ is aperiodic, φ is injective on G. The 1 : 1 correspondence between G and any $\mathscr{H}$-class in J discussed in (4.3) establishes the assertion.

(b) Let J be a regular $\mathscr{J}$-class of M, and suppose that φ does not separate J and $B(J)$. Then there are elements $m \in J$, $m' \in B(J)$ such that $m\varphi = m'\varphi$. Since J is regular, there is an idempotent $e \in J$ with $e\mathscr{R}m$. Thus, we can write $e = ma$ for some $a \in M$. Consider the element $em'ae \in B(J)$. Since e is an idempotent, we see that $(em'ae)\varphi = e\varphi$. Let f be the idempotent that results from raising $em'ae$ to a sufficiently high power. Then clearly, $f\varphi = e\varphi$ and $\{e, f\}$ is a copy of U_1 in M. Therefore, φ is not U_1-free. □

Proposition 4.4. *Let $\varphi : M \to N$ be a $\mathscr{J}$-singular morphism of monoids. If φ is both aperiodic and U_1-free, then φ is locally trivial.*

Proof. Let $K_\varphi(n_L, n_R)$ be a local monoid of K_φ, and let $(m,n) \in \#\varphi$ with $n_L n = n_L$

and $nn_R = n_R$. To show that $K_\varphi(n_L, n_R)$ is trivial, we must show that (m, n) and $(1, 1)$ define the same function (1.3). Equivalently, we must show

(4.9) $\qquad amb = ab$

for all $a \in n_L\varphi^{-1}$ and $b \in n_R\varphi^{-1}$.

Let $a \in n_L\varphi^{-1}$ and $b \in n_R\varphi^{-1}$. The equations $n_L = n_L n$ and $n_R = nn_R$ establish

(4.10) $\qquad a\varphi = (am)\varphi \quad \text{and} \quad b\varphi = (mb)\varphi$

from which follows the equation

(4.11) $\qquad (ab)\varphi = (amb)\varphi.$

Suppose $a \in A(J)$. Since φ separates $A(J)$ and J, (4.10) implies that $am \in A(J)$. Since φ is injective on $A(J)$, we conclude that $a = am$, from which (4.9) follows. If $a \in B(J)$, then since $B(J)$ is an ideal, we have $am \in B(J)$. But φ is injective on $B(J)$, so again $a = am$. We have shown that if $a \in M - J$, then (4.9) holds. A dual argument proves that if $b \in M - J$, then (4.9) holds. Finally, if both ab and amb belong to $M - J$, (4.11) implies (4.9).

Therefore, we may assume $a, b \in J$, and either ab or amb belongs to J. It follows that J must be a regular $\mathcal{J}$-class. Since φ is U_1-free, Lemma 4.3(b) states that φ separates J and $B(J)$. It follows that both ab and amb belong to J. We show that $ab\mathcal{H}amb$. First we have $a \geq ab$ and $a \geq amb$ in the $\mathcal{R}$-ordering of M. But since $a, ab, amb \in J$, it follows that $ab\mathcal{R}a\mathcal{R}amb$. A dual argument shows that $ab\mathcal{L}abm$.

Now since $(ab)\varphi = (amb)\varphi$ and $ab\mathcal{H}amb$, Lemma 4.3(a) shows that $ab = amb$. Therefore, (4.9) holds in all cases, so φ is locally trivial. □

We can now prove the first part of Proposition 3.7. Recall the statement: If an MPS $\varphi : M \to N$ is U_1-free, then φ is primitive.

Proof of Proposition 3.7(a). The assertion follows directly from Propositions 4.2 and 4.4 according to whether or not φ is aperiodic. If φ is aperiodic, then φ satiesfies the hypothesis of Proposition 4.4, and hence is locally trivial. If φ is not aperiodic, then by Proposition 4.2, φ is primitive of type Q, where Q is a simple group. □

Before presenting the proof of Proposition 3.7(b), we state some other interesting corollaries of Proposition 4.4. If $\varphi : M \to N$ is $\mathcal{J}$-singular and one of its singular $\mathcal{J}$-classes is null, then φ must be both aperiodic and U_1-free. This follows from the fact that if $f \in N$ is an idempotent, then $f\varphi^{-1}$ can contain at most one regular element of M. This proves

Corollary 4.5. *If $\varphi : M \to N$ is a $\mathcal{J}$-singular morphism with a singular $\mathcal{J}$-class that is null, then φ is locally trivial.* □

Corollary 4.6. *Let M have a 0-minimal null ideal N, and let $\eta : M \to M/N$ be the quotient. Then η is locally trivial.* □

Example 4.7. This example shows that the kernel K_φ cannot be replaced in Proposition 4.4 by the derived category D_φ. Let M be the cyclic monoid defined by $x^3 = x^2$. Thus $M = \{1, x, x^2\}$. Consider the morphism $\varphi : M \to U_1$ defined by $x\theta = 0$. φ is clearly an MPS that is both aperiodic and U_1-free, so by Proposition 4.4, $K_\varphi \in \ell\mathbf{1}$. The derived category D_φ has two local monoids, $D_\varphi(1)$ and $D_\varphi(0)$, and a direct calculation shows that $D_\varphi(0) \approx U_1$. Hence D_φ does not belong to $\ell\mathbf{1}$.

We now present a proof of Proposition 3.7(b), which we restate as a separate proposition.

Proposition 4.8. *Let $\varphi : M \to N$ be an MPS that is not U_1-free. Then φ has a decomposition*

$$\varphi = \theta\psi$$

where θ is primitive of type U_1, and ψ is a locally trivial morphism.

Proof. An MPS must be $\mathscr{J}$-singular, so let J be a singular $\mathscr{J}$-class of φ. Select a monoid T with the property that $T - \{1\}$ is in 1 : 1 correspondence with the $\mathscr{R}$-classes in J. Choose a fixed correspondence, and denote the $\mathscr{R}$-class corresponding to each $a \in T - 1$ by R_a.

A second piece of needed notation is the following: For each $n \in A(J)\varphi$, let $\bar{n}$ denote the unique member of $A(J)$ satisfying $\bar{n}\varphi = n$. This notation is possible because φ is injective on $A(J)$.

We construct the decomposition $\varphi = \theta\psi$ by means of a wreath product. Define a function

$$h : M \to T \circ N, \qquad mh = (f_m, m\varphi)$$

where the function $f_m : N \to T$ is given by

$$nf_m = \begin{cases} a & \text{if } n \in A(J)\varphi,\ \bar{n}m \in R_a, \\ 1 & \text{otherwise.} \end{cases}$$

Let $\theta : M \to T \circ N$ be the relation of monoids generated by the function h. In other words, if $m \in M$,

$$m\theta = \{m_1h \dots m_kh\text{: for each factorization } m = m_1 \dots m_k\}.$$

Thus, if $m \in M$ and $(f, n) \in m\theta$, then there is a factorization $m = m_1 \dots m_k$ so that $(f, n) = m_1h \dots m_kh$. Denoting m_ih by $(f_i, m_i\varphi)$, we deduce that $n = m\varphi$ and

$$(4.12) \qquad f = [1, f_1] + [m_1\varphi, f_2] + \cdots + [(m_1 \dots m_{k-1})\varphi, f_k].$$

Here, the notation $[n, f]$ denotes the left action of n on the function f used to define the wreath product. Recall that the result is a function $[n, f] : N \to T$ given by $n'[n, f] = (n'n)f$.

Next, define $\psi : M\theta \to N$ to be the projection $\pi : T \circ N \to N$ restricted to $M\theta$. Since $(f, n) \in m\theta$ implies that $n = m\varphi$, it follows that $\varphi = \theta\psi$.

We first establish that ψ is locally trivial by showing that $D_\psi \in \ell\mathbf{1}$. Let $n_L \in N$ and let $(f,n) \in M\theta$ with $n_L n = n_L$. In order to show that the local monoid $D_\psi(n_L)$ is trivial, we must show that right multiplication by (f,n) on the set $n_L\psi^{-1}$ is the identity map. That is, we must show that $(f_L, n_L)(f,n) = (f_L, n_L)$ for each $(f_L, n_L) \in M\theta$. Since we have

$$(f_L, n_L)(f,n) = (f_L + [n_L, f], n_L n) \quad \text{and} \quad n_L n = n_L,$$

it suffices to show that the function $[n_L, f]$ is the identity function. That is, it suffices to show that $(n_0 n_L)f = 1$ for all $n_0 \in N$.

Since $(f,n) \in m\theta$ for some $m \in M$, we may assume that f is given by the equation (4.12) associated with a factorization $m = m_1 \dots m_k$. Let $n_i = m_i\varphi$, $i = 1, \dots, k$. Then $n = n_1 \dots n_k$, and

$$(n_0 n_L)f = (n_0 n_L)f_1 + (n_0 n_L n_1)f_2 + \cdots + (n_0 n_L n_1 \dots n_{k-1})f_k.$$

Suppose that $(n_0 n_L)f \neq 1$. Then $(n_0 n_L n_1 \dots n_{i-1})f_i \neq 1$ for some $i = 1, \dots, n$. This means that $n_0 n_L n_1 \dots n_{i-1} \in A(J)\varphi$ and $n_0 n_L n_1 \dots n_i \in J\varphi$. Since φ separates $A(J)$ and J, and since $J \cup B(J)$ is an ideal of M, it follows that the complement of $A(J)\varphi$ is an ideal of N. This leads to the conclusion that $n_0 n_L n_1 \dots n_{i-1}$ and $n_0 n_L n_1 \dots n_i$ are not $\mathscr{R}$-equivalent. However, the calculation

$$n_0 n_L = n_0 n_L n = n_0 n_L n_1 \dots n_k$$

shows otherwise. Thus we have arrived at a contradiction. It must be that $[n_L, f]$ is the identity function. Therefore, ψ is a locally trivial morphism.

We now treat the relation θ. We first note that θ is not U_1-free. For if θ were U_1-free, then since ψ is U_1-free, the composition $\theta\psi = \varphi$ would be U_1-free, contrary to the hypothesis. To prove that θ is primitive of type U_1, it remains to show that $K_\theta \in (U_1)$. We first prove that θ is a $\mathscr{J}$-singular relation that is injective on the $\mathscr{L}$-classes of M.

Because φ is $\mathscr{J}$-singular and because $\varphi = \theta\psi$, it follows easily that θ is a $\mathscr{J}$-singular relation with J as a singular $\mathscr{J}$-class. Since θ is injective on $M - J$, it suffices to show that θ is injective on the $\mathscr{L}$-classes of J. Let $m \in R_a \subseteq J$, and let $(f,n) \in m\theta$. We claim that $1f = a$. To show this, we may assume that m has a factorization $m = m_1 \dots m_k$ and that f is given by (4.12). Let j be the unique integer between 1 and k satisfying

$$m_1 \dots m_{j-1} \in A(J) \quad \text{and} \quad m_1 \dots m_j \in J.$$

Since $m \in J$ and $m = (m_1 \dots m_j)m_{j+1} \dots m_k$, it follows that $m_1 \dots m_j \mathscr{R} m$. Therefore, $(m_1 \dots m_{j-1})\varphi f_j = a$, and since $(m_1 \dots m_{i-1})\varphi f_i = 1$ for each $i \neq j$, we obtain $1f = a$.

Now assume $m \mathscr{L} m' \in J$ with $(f,n) \in m\theta \cap m'\theta$. Then $m\varphi = m'\varphi$, and m and m' belong to the same $\mathscr{R}$-class, namely R_{1f}. Therefore, $m \mathscr{H} m'$. Since φ is not U_1-free, it follows that φ does not separate $\mathscr{H}$-classes of M. Therefore, by (4.2), φ is injective on $\mathscr{H}$-classes of M. We may conclude that $m = m'$. Therefore, θ is injective on $\mathscr{L}$-classes of M.

That θ is primitive of type U_1 will now follow from the next proposition.

Proposition 4.9. *Let $\varphi: M \to N$ be a $\mathscr{J}$-singular relation of monoids. If φ is either injective on $\mathscr{L}$-classes of M or injective on $\mathscr{R}$-classes of M, then*

$$K_\varphi \in (U_1).$$

Proof. We shall assume that φ is injective on $\mathscr{R}$-classes of M and prove $D_\varphi \in (U_1)$. Since $K_\varphi < D_\varphi$, this establishes the assertion in this case. The $\mathscr{L}$-class case then follows as a corollary. For, if φ is injective on $\mathscr{L}$-classes of M, then φ^ϱ is injective on $\mathscr{R}$-classes of M^ϱ. It follows from the first assertion that $K_{\varphi^\varrho} \in (U_1)$. But (U_1) is closed under reversal, so using Proposition 1.6, we have $K_\varphi \approx (K_{\varphi^\varrho})^\varrho \in (U_1)$.

Therefore, let us assume that φ is injective on $\mathscr{R}$-classes of M. Theorem 2.9 states that we only need to show that the local monoids of D_φ belong to (U_1). Monoids in (U_1) are defined by the equations

(4.13) $\quad x^2 = x, \qquad xy = yx.$

Hence, it suffices to show that every local monoid of D_φ satisfies (4.13). Let $n_L \in N$, and let $(m,n),(m',n') \in \#\varphi$ with $n_L n = n_L = n_L n'$. Then the functions

$$[m,n],[m',n'] : n_L\varphi^{-1} \to n_L\varphi^{-1}$$

where $a[m,n] = am$, must satisfy (4.13). In other words, we must establish

(4.14) $\quad am^2 = am \quad$ and $\quad amm' = am'm$

for all $a \in n_L\varphi^{-1}$.

We first note that if $a, a' \in n_L\varphi^{-1}$ and $a, a' \in M - J$, then $a = a'$. For φ is injective on $M - J$ and $a\varphi \cap a'\varphi \neq \emptyset$. Second, if $a \in n_L\varphi^{-1}$ and $(m,n) \in \#\varphi$ with $n_L n = n_L$, then

(4.15) $\quad$ Either $am = a$ or $a \in J$, $am \in B(J)$.

For the proof, note that $n_L = n_L n \in (am)\varphi$, so $a\varphi \cap (am)\varphi \neq \emptyset$. If $a \in A(J)$, then since φ separates $A(J)$ and J, $am \in A(J)$. Therefore $a = am$. If a belongs to the ideal $B(J)$, then so does am, so again $a = am$. Last, if $a \in J$, then either $am \in J$ or $am \in B(J)$. If $am \in J$, then $a\mathscr{R}am$. Since φ is injective on $\mathscr{R}$-classes, we have $a = am$. This establishes (4.15).

We now establish (4.14). If $a = am$, then $am = am^2$. Otherwise, $am \in B(J)$ in which case am^2 does also. Since $am, am^2 \in n_L\varphi^{-1} \cap M - J$, we conclude that $am = am^2$ in all cases.

If $a = am = am'$, then $amm' = am'm$. If $a \in J$ and $am \in B(J)$, then $amm' \in B(J)$. Either $am' = a$ or $am' \in B(J)$. In either case, $am'm \in B(J)$. Since $amm', am'm \in n_L\varphi^{-1} \cap M - J$, we obtain $amm' = am'm$ in all cases. This establishes (4.14) and proves the assertion. $\square$

With the conclusion of the proof of Proposition 4.9, we have also established Proposition 3.7 and hence, Theorem 3.1.

5. Kernel properties

We now lift our cardinality restrictions, and again treat monoids and categories of arbitrary cardinality. We will not restrict our scope again. In this section we develop the relationship between the kernel and certain compositions of relations.

Proposition 5.1. *In the commuting diagram of relations*

$$\begin{array}{ccc} M & \xrightarrow{\theta} & M' \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi'} & \\ & N & \end{array}$$

of monoids, assume that θ is injective. Then $K_\varphi < K_{\varphi'}$.

Proof. We show that $\theta : M \to M'$ induces a relational morphism of categories

$$\theta' : W_\varphi \to K_{\varphi'},$$

$$n\theta' = n \quad \text{on objects,}$$

$$(n_{\rm L}, (m, n), n'_{\rm R})\theta' = \{[n_{\rm L}, (m', n), n'_{\rm R}] : m' \in m\theta\}.$$

Since $\varphi = \theta\varphi'$, it follows that $M\varphi = M\theta\varphi' \subseteq M'\varphi'$. Thus the object function of θ' is well defined. Let $(m, n) \in \#\varphi$. Then $n \in m\varphi = m\theta\varphi'$, so there exists $m' \in m\theta$ such that $(m', n) \in \#\varphi'$. Therefore, θ' on hom-sets is both well defined and fully defined. Straightforward calculations now show that θ' is a relational morphism.

Composing $\eta^{-1} : K_\varphi \to W_\varphi$ with θ', we obtain the relational morphism

$$\eta^{-1}\theta' : K_\varphi \triangleleft K_{\phi'}.$$

Showing that $\eta^{-1}\theta'$ is injective on hom-sets of K_φ will establish the assertion. To this end, let $(m, n), (\bar{m}, \bar{n}) \in W_\varphi(n, n')$, and suppose there exist $m' \in m\theta$ and $\bar{m}' \in \bar{m}\theta$ with

$$[n_{\rm L}, (m', n), n'_{\rm R}] = [{\rm n}_{\rm L}, (\bar{m}', \bar{n}), n'_{\rm R}]$$

in $K_{\varphi'}$. It must be shown that $(m, n)\eta = (\bar{m}, \bar{n})\eta$. By (1.3), this equality is established by showing that $m_{\rm L} m m'_{\rm R} = {\rm m}_{\rm L} \bar{m} m'_{\rm R}$ for all $m_{\rm L} \in n_{\rm L}\varphi^{-1}$ and $m'_{\rm R} \in n'_{\rm R}\varphi^{-1}$.

Let $m_{\rm L} \in n_{\rm L}\varphi^{-1}$, and using the fact that $\varphi = \theta\varphi'$, choose $m''_{\rm L} \in m_{\rm L}\theta$ so that $n_{\rm L} \in m''_{\rm L}\varphi'$. Similarly, let $m'_{\rm R} \in n'_{\rm R}\varphi^{-1}$, and choose $m''_{\rm R} \in m'_{\rm R}\theta$ so that $n'_{\rm R} \in m''_{\rm R}\varphi'$. Then

$$m''_{\rm L} m' m''_{\rm R} \in m_{\rm L}\theta m\theta m'_{\rm R}\theta \subseteq (m_{\rm L} m m'_{\rm R})\theta$$

and

$$m''_{\rm L} \bar{m}' m''_{\rm R} \in m_{\rm L}\theta \bar{m}\theta m'_{\rm R}\theta \subseteq (m_{\rm L} \bar{m} m'_{\rm R})\theta.$$

However, since $[n_{\rm L}, (m', n), n'_{\rm R}] = [n_{\rm L}, (\bar{m}', \bar{n}), n'_{\rm R}]$ and because $m''_{\rm L} \in n_{\rm L}\varphi^{-1}$ and $m''_{\rm R} \in n'_{\rm R}\varphi^{-1}$, we have $m''_{\rm L} m' m''_{\rm R} = m''_{\rm L} \bar{m}' m''_{\rm R}$. Therefore

$$(m_{\rm L} m m'_{\rm R})\theta \cap (m_{\rm L} \bar{m} m'_{\rm R})\theta \neq \emptyset$$

and since θ is injective, this implies that $m_L m m'_R = m_L \bar{m} m'_R$. Therefore, $(m,n)\eta = (\bar{m},\bar{n})\eta$, and $K_\varphi < K_{\varphi'}$. □

Let $\varphi, \varphi' : M \to N$ be relations. We write $\varphi \subseteq \varphi'$ if $m\varphi \subseteq m\varphi'$ for all $m \in M$. Equivalently, $\varphi \subseteq \varphi'$ iff $\#\varphi$ is a submonoid of $\#\varphi'$. Evidently then, $\varphi \subseteq \varphi'$ iff $\varphi^{-1} \subseteq \varphi'^{-1}$.

Proposition 5.2. *If $\varphi, \varphi' : M \to N$ are relations of monoids and $\varphi \subseteq \varphi'$, then*

$$K_\varphi < K_{\varphi'}.$$

Proof. Let $(\alpha, \#\varphi, \beta)$ and $(\alpha', \#\varphi', \beta')$ be, respectively, the canonical factorizations of φ and φ', and let $j : \#\varphi \to \#\varphi'$ be the inclusion morphism. Then

$$\beta = j\beta' : \#\varphi \to N.$$

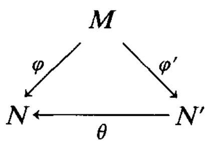
Since j is injective, we deduce from Proposition 5.1 that

$$K_\beta < K_{\beta'}.$$

The assertion now follows from Proposition 1.8. □

As a dual result to Proposition 5.1, we present

Proposition 5.3. *In the commuting diagram of relations*



of monoids, assume that θ is injective. Then $K_\varphi < K_{\varphi'}$.

Proof. Let ψ denote $\theta^{-1} : N \to N'$. Since θ is an injective relation, ψ is a partial function and $\theta\psi \subseteq I_{N'}$. Therefore, the assumption $\varphi = \varphi'\theta$ implies the containment $\varphi\psi \subseteq \varphi'$. We show that ψ induces a morphism of categories

$$\psi' : W_\varphi \to K_{\varphi'},$$

$$n\psi' = (n_L\psi, n_R\psi) \quad \text{on objects},$$

$$(n_L, (m,n), n'_R)\psi' = [n_L\psi, (m, n\psi), n'_R\psi].$$

Since $\varphi = \varphi'\theta$, we see that $M\varphi \subseteq \text{rg}(\theta) = \text{dom}\,\psi$. Therefore for each $n \in M\varphi$, we have $\emptyset \neq n\psi \in M\varphi\psi \subseteq M\varphi'$. Thus ψ' is a well-defined function on objects of W_φ. Let $(n_L, (m,n), n'_R)$ be an arrow of W_φ. Then $n\psi \in m\varphi\psi \subseteq m\varphi'$, so $(m, n\psi) \in \#\varphi'$. Since $n_L n = n'_L$, we have $n_L\psi n\psi = n'_L\psi$; dually, $n_R\psi = n\psi n'_R\psi$. This information shows that the hom-set functions of ψ' are well defined. This established, the fact that ψ' is a morphism follows easily.

Now composing $\eta^{-1}: K_\varphi \to W_\varphi$ with ψ' results in a relational morphism

$$\eta^{-1}\psi : K_\varphi \lhd K_{\varphi'}.$$

Showing that $\eta^{-1}\psi$ is injective on hom-sets of K_φ establishes the assertion. To do so, let $(m,n),(m',n') \in W_\varphi(n,n')$ and suppose that $(m,n)\psi' = (m',n')\psi'$. We must show that $(m,n)\eta = (m',n')\eta$. Let $m_{\mathrm{L}} \in n_{\mathrm{L}}\varphi^{-1}$ and $m'_{\mathrm{R}} \in n'_{\mathrm{R}}\varphi^{-1}$. Then

$$m_{\mathrm{L}} \in n_{\mathrm{L}}\varphi^{-1} = n_{\mathrm{L}}(\varphi'\theta)^{-1} = (n_{\mathrm{L}}\psi)\varphi'^{-1}.$$

Similarly, $m'_{\mathrm{R}} \in (n'_{\mathrm{R}}\psi)\varphi^{-1}$. The assumption

$$[n_{\mathrm{L}}\psi, (m, n\psi), n'_{\mathrm{R}}\psi] = [n_{\mathrm{L}}\psi, (m', n'\psi), n'_{\mathrm{R}}\psi]$$

implies that $m_{\mathrm{L}} m m'_{\mathrm{R}} = m_{\mathrm{L}} m' m'_{\mathrm{R}}$. This in turn implies that $(m,n)\eta = (m',n')\eta$ and establishes the assertion. □

Combining Propositions 5.1 and 5.3, we obtain

Corollary 5.4. *If θ and θ' are injective relations, then*

$$K_{\theta\varphi\theta'} < K_\varphi. \qquad \square$$

Another useful corollary of Proposition 5.3 is the following:

Proposition 5.5. *Let $\varphi : M \to N$ be a relation of monoids and let $\theta : N \to N'$ be a morphism. Then $K_\varphi < K_{\varphi\theta}$.*

Proof. Let $\varphi' = \varphi\theta$. Then since θ is a function, we have $I_{N'} \subseteq \theta\theta^{-1}$. Therefore,

$$\varphi \subseteq \varphi\theta\theta^{-1} = \varphi'\theta^{-1}.$$

Since θ^{-1} is injective, we may apply Propositions 5.2 and 5.3 to obtain

$$K_\varphi < K_{\varphi'\theta^{-1}} < K_{\varphi'}. \qquad \square$$

6. The double semidirect product

We introduce in this section a 'two-sided' version of the semidirect product of monoids, which we call the double semidirect product.

Let V and T be monoids. To aid notational clarity, we will write V additively and let 0 denote its identity; however, commutivity for V is not assumed. A *double action* of T on V is a function

(6.1) $\qquad T \times V \times T \to V, \qquad (t, v, t') \to tvt'$

satisfying the following conditions:

$$(6.2)\qquad \begin{aligned} &t(v_1+v_2)t'=tv_1t'+tv_2t', \qquad t_1(t_2vt_2')=(t_1t_2)v(t_2't_1'),\\ &1v1=v, \qquad t0t'=0 \end{aligned}$$

for all $t, t', t_1, t_2, t_1', t_2' \in T$ and all $v, v_1, v_2 \in V$.

Given a double action of T on V, the associated *double semidirect product* $V{**}T$ is the set $V\times T$ equipped with the multiplication rule

$$(v,t)(v',t')=(1vt'+tv'1, tt').$$

Conditions (6.2) guarantee that this product is associative and that $V{**}T$ is a monoid with identity $(0,1)$. Indeed, for associativity we have

$$\begin{aligned} ((v,t)(v',t'))(v'',t'')&=(1vt'+tv'1, tt')(v'',t'')\\ &=(1(1vt'+tv'1)t''+tt'v''1, tt't'') \end{aligned}$$

while

$$\begin{aligned} (v,t)((v',t')(v'',t''))&=(v,t)(1v't''+t'v''1, t't'')\\ &=(1vt't''+t(1v't''+t'v''1)1, tt't''). \end{aligned}$$

Applying the first two conditions of (6.2) to these results shows that both computations equal

$$(1vt't''+tv't''+tt'v''1, tt't'').$$

A direct calculation using the last two conditions of (6.2) shows that $(0,1)$ is the identity of $V{**}T$. Furthermore, if T and V are groups, then $V{**}T$ is a group. The inverse of (v,t) is $(-(t^{-1}vt^{-1}), t^{-1})$.

Because of the rule $1v1=v$, we may simplify notation in certain case. The expression $tv1$ will usually be written tv, and the expression $1tv$ will be replaced by tv. With these conventions, the following one-sided version of conditions (6.2), as well as their duals, hold:

$$\begin{aligned} &t(v_1+v_2)=tv_1+tv_2, \qquad t_1(t_2v)=(t_1t_2)v,\\ &1v=v, \qquad t0=0. \end{aligned}$$

In addition, the relationship $(tv)t'=t(vt')$ holds. Thus a double action of T on V can be viewed as a left action and a right action of T on V which are associative.

With this notational convention, the multiplication in the double semidirect product may be written

$$(v,t)(v',t')=(vt'+tv', tt').$$

It is also instructive to write out a product of n terms of $V{**}T$:

$$\begin{aligned} (v_1,t_1)\cdots(v_n,t_n)=(v_1(t_2\ldots t_n)+\cdots+(t_1\ldots t_{k-1})v_k(t_{k+1}\ldots t_n)+\cdots\\ +(t_1\ldots t_{n-1})v_n, t_1\ldots t_n). \end{aligned}$$

If the double action of T on V does not depend on the right action, that is,

$$tvt' = tv \quad \forall t, t' \in T \text{ and } v \in V,$$

then the double action is a left action of T on V, $(t, v) \to tv$. In this case the double semidirect product is an ordinary semidirect product $V * T$. Dually, if the double action is a right action, then $V ** T$ is a reverse semidirect product $V *_{\varrho} T$. If the double action is the identity action, that is, $tvt' = v$ for all $t, t' \in T$ and $v \in V$, then the double semidirect product is the direct product $V \times T$.

Therefore, every direct product, every semidirect product, and every reverse semidirect product is a double semidirect product. There are double semidirect products that cannot be characterized as semidirect products. However, in the case when the right-hand term is a group, any double semidirect product of the form $V ** G$ is actually a semidirect product.

Example 6.1. If G is a group, and there is a given double action of G on a monoid V, then the resulting product $V ** G$ is isomorphic to a semidirect product $V * G$. To show this, assume that a double action of G on V is given, and define a left action of G on V by

$$(g, v) \to gvg^{-1}.$$

Then $(v, g) \to (vg^{-1}, g)$ defines an isomorphism $V ** G \to V * G$.

Because the multiplication rule for the double semidirect product behaves like the direct product in the right-hand coordinate, the projection function $\pi : V \times T \to T$ is a morphism

$$\pi : V ** T \to T$$

of monoids. The projection onto V does not enjoy this status.

If either V or T is the trivial monoid, then there is only one possible double action of T on V. In these cases, we have

$$(6.3) \qquad 1 ** T \approx T \quad \text{and} \quad V ** 1 \approx V.$$

Both the injections

$$\alpha : V \to V ** T, \qquad v\alpha = (v, 1),$$

$$\beta : T \to V ** T, \qquad t\beta = (0, t)$$

are morphisms. Therefore

$$(6.4) \qquad V \prec V ** T \quad \text{and} \quad T \prec V ** T.$$

Given a double semidirect product $V ** T$, we may define a double action of T^{ϱ} on V^{ϱ} by

$$(t, v, t') \to t'vt.$$

A routine calculation shows that with this double action,

$$(6.5) \qquad V^{\varrho} ** T^{\varrho} \approx (V ** T)^{\varrho}.$$

The triple product (M, V, N), introduced in [2, p. 142], is a double semidirect product. In that definition there is given a left action of N on V and a right action of M on V, and these actions are associative. That is,

$$(nv)m = n(vm).$$

The underlying set of (M, V, N) is the direct product $M \times V \times N$, and the multiplication is given by

$$(m, v, n)(m', v', n') = (mm', vm' + nv', nn').$$

Using these two actions, we may define a double action of $M \times N$ on V by

$$(m, n)v(m', n') = (nv)m'.$$

This double action defines a double semidirect product $V**(M \times N)$ which is easily seen to be isomorphic to the triple product (M, V, N).

We now present the central result of this section, which details an important relationship between the kernel and the double semidirect product.

Let $\varphi : M \to N$ be a relation of monoids, and let T be a monoid. Suppose there are double actions of T on M and T on N. This pair of actions is *compatible with* φ if

$$(m, n) \in \#\varphi \quad \Rightarrow \quad (tmt', tnt') \in \#\varphi \quad \forall t, t' \in T.$$

In this situation, $\varphi : M \to N$ induces a relation of monoids

$$\varphi ** T : M ** T \to N ** T, \qquad (m, t)\varphi ** T = \{(n, t) : n \in m\varphi\}$$

as the reader may easily verify.

Theorem 6.2. *Let $\varphi : M \to N$ be a relation of monoids, and let T be a monoid. Suppose there are double actions of T on M and T on N that are compatible with φ. Then*

$$K_{\varphi ** T} \sim K_\varphi.$$

Proof. Let $\alpha_M : M \to M ** T$ be the injection $m \to (m, 1)$, and let $\alpha_N : N \to N ** T$ be defined similarly. Then we may write $\varphi = \alpha_M \varphi ** T \alpha_N^{-1}$. Since both α_M and a_N^{-1} are injective relations of monoids, it follows from Corollary 5.4 that $K_\varphi < K_{\varphi ** T}$.

For the opposite inequality, we let $\eta : W_{\varphi ** T} \to K_{\varphi ** T}$ be the two-stage construction of $K_{\varphi ** T}$. We will define a morphism

$$\theta : W_{\varphi ** T} \to K_\varphi$$

and then show that the relational morphism $\eta^{-1}\theta : K_{\varphi ** T} \to K_\varphi$ is a division.

Note that $(M ** T)\varphi ** T = M\varphi \times T$; therefore,

$$\mathrm{Obj}(W_{\varphi ** T}) = (M\varphi \times T) \times (M\varphi \times T).$$

Define the object function

$$\theta : \mathrm{Obj}(W_{\varphi ** T}) \to \mathrm{Obj}(K_\varphi), \qquad ((n_L, t_L), (n_R, t_R))\theta = (n_L t_R, t_L n_R).$$

Using the boldface notation for object pairs introduced in Section 1, we may rewrite the above by

$$(\boldsymbol{n},\boldsymbol{t})\theta=(n_{\rm L}t_{\rm R},t_{\rm L}n_{\rm R})$$

when we interpret $(\boldsymbol{n},\boldsymbol{t})$ as $((n_{\rm L},t_{\rm L}),(n_{\rm R},t_{\rm R}))$. This definition is well defined, for if $n\in M\varphi$ and $t\in T$, then $(m,n)\in\#\varphi$ for some $m\in M$, and both (mt,nt) and (tm,tn) belong to $\#\varphi$. In particular, both $n_{\rm L}t_{\rm R}$ and $t_{\rm L}n_{\rm R}$ belong to $M\varphi$, so $(n_{\rm L}t_{\rm R},t_{\rm L}n_{\rm R})\in{\rm Obj}(K_\varphi)$.

Define the hom-set functions by

$$\theta: W_{\varphi**T}((\boldsymbol{n},\boldsymbol{t}),(\boldsymbol{n}',\boldsymbol{t}'))\to K_\varphi((n_{\rm L}t_{\rm R},t_{\rm L}n_{\rm R}),(n'_{\rm L}t'_{\rm R},t'_{\rm L}n'_{\rm R})),$$

$$\{(m,t),(n,t)\}\theta=[n_{\rm L}t_{\rm R},(t_{\rm L}mt'_{\rm R},t_{\rm L}nt'_{\rm R}),t'_{\rm L}n'_{\rm R}].$$

Note that $(n_{\rm L},t_{\rm L})(n,t)=(n'_{\rm L},t'_{\rm L})$ and $(n_{\rm R},t_{\rm R})=(n,t)(n'_{\rm R},t'_{\rm R})$. Expanding these equations, we obtain

$$n'_{\rm L}=n_{\rm L}t+t_{\rm L}n,\qquad n_{\rm R}=nt'_{\rm R}+tn'_{\rm R},$$

$$t'_{\rm L}=t_{\rm L}t,\qquad t_{\rm R}=tt'_{\rm R}.$$

It follows that

$$n_{\rm L}t_{\rm R}+t_{\rm L}nt'_{\rm R}=n_{\rm L}tt'_{\rm R}+t_{\rm L}nt'_{\rm R}=(n_{\rm L}t+t_{\rm L}n)t'_{\rm R}=n'_{\rm L}t'_{\rm R}$$

and

$$t_{\rm L}nt'_{\rm R}+t'_{\rm L}n'_{\rm R}=t_{\rm L}nt'_{\rm R}+t_{\rm L}tn'_{\rm R}=t_{\rm L}(nt'_{\rm R}+tn'_{\rm R})=t_{\rm L}n_{\rm R}.$$

These calculations show that θ is well defined.

The identity arrow at object $(\boldsymbol{n},\boldsymbol{t})$ is $\{(0,1),(0,1)\}$, and

$$\{(0,1),(0,1)\}\theta=[n_{\rm L}t_{\rm R},(t_{\rm L}0t_{\rm R},t_{\rm L}0t_{\rm R}),t_{\rm L}n_{\rm R}]=[n_{\rm L}t_{\rm R},(0,0),t_{\rm L}n_{\rm R}]$$

which is the identity arrow of K_φ at object $(\boldsymbol{n},\boldsymbol{t})\theta$.

Let

$$\{(m,t)(n,t)\}:(\boldsymbol{n},\boldsymbol{t})\to(\boldsymbol{n}',\boldsymbol{t}'),\qquad \{(m',t')(n',t')\}:(\boldsymbol{n}',\boldsymbol{t}')\to(\boldsymbol{n}'',\boldsymbol{t}'')$$

be consecutive arrows in $W_{\varphi**T}$. Then using $t'_{\rm L}=t_{\rm L}t$ and $t'_{\rm R}=t't''_{\rm R}$, we have

$$\begin{aligned}
&\{(m,t)(n,t)\}\theta\{(m',t')(n',t')\}\theta\\
&=[n_{\rm L}t_{\rm R},(t_{\rm L}mt'_{\rm R},t_{\rm L}nt'_{\rm R}),t'_{\rm L}{\rm n}'_{\rm R}][n'_{\rm L}t'_{\rm R},(t'_{\rm L}m't''_{\rm R},t'_{\rm L}n't''_{\rm R}),t''_{\rm L}n''_{\rm R}]\\
&=[n_{\rm L}t_{\rm R},(t_Lmt'_{\rm R}+t'_{\rm L}m't''_{\rm R},t_{\rm L}nt'_{\rm R}+t'_{\rm L}n't''_{\rm R}),t''_{\rm L}n''_{\rm R}]\\
&=[n_{\rm L}t_{\rm R},(t_{\rm L}mt't''_{\rm R}+t_{\rm L}tm't''_{\rm R},t_{\rm L}nt't''_{\rm R}+t_{\rm L}tn't''_{\rm R}),t''_{\rm L}n''_{\rm R}]\\
&=[n_{\rm L}t_{\rm R},(t_{\rm L}(mt'+tm')t''_{\rm R},t_{\rm L}(nt'+tn')t''_{\rm R}),t''_{\rm L}n''_{\rm R}]\\
&=\{(mt'+tm',tt'),(nt'+tn',tt')\}\theta\\
&=(\{(m,t)(n,t)\}\{(m',t')(n',t')\})\theta.
\end{aligned}$$

Therefore, θ is a morphism of categories.

To show that $\eta^{-1}\theta : K_{\varphi ** T} \lhd K_\varphi$ is a division, it suffices to suppose that

(6.6) $$\{(m,t),(n,t)\}\theta = \{(m',t'),(n',t')\}\theta$$

where $\{(m,t),(n,t)\}$ and $\{(m',t'),(n',t')\}$ are coterminal arrows belonging to $W_{\varphi ** T}((n,t),(n',t'))$, and to show that

(6.7) $$(m_\mathrm{L}, t_\mathrm{L})(m,t)(m'_\mathrm{R}, t'_\mathrm{R}) = (m_\mathrm{L}, t_\mathrm{L})(m',t')(m'_\mathrm{R}, t'_\mathrm{R})$$

for all $m_\mathrm{L} \in n_\mathrm{L}\varphi^{-1}$ and $m'_\mathrm{R} \in n'_\mathrm{R}\varphi^{-1}$. Equation (6.7) may be rewritten

$$[n_\mathrm{L}t_\mathrm{R}, (t_\mathrm{L}mt'_\mathrm{R}, t_\mathrm{L}nt'_\mathrm{R}), t'_\mathrm{L}n'_\mathrm{R}] = [n_\mathrm{L}t_\mathrm{R}, (t_\mathrm{L}m't'_\mathrm{R}, t_\mathrm{L}n't'_\mathrm{R}), t'_\mathrm{L}n'_\mathrm{R}]$$

and is satisfied if and only if

(6.8) $$\bar{m} + t_\mathrm{L}mt'_\mathrm{R} + \bar{m}' = \bar{m} + t_\mathrm{L}m't'_\mathrm{R} + \bar{m}'$$

for all $\bar{m} \in (n_\mathrm{L}t_\mathrm{R})\varphi^{-1}$ and $m' \in (t'_\mathrm{L}n'_\mathrm{R})\varphi^{-1}$.

The left-hand side of (6.7) expands to

$$(m_\mathrm{L}tt'_\mathrm{R} + t_\mathrm{L}mt'_\mathrm{R} + t_\mathrm{L}tm'_\mathrm{R}, t_\mathrm{L}tt'_\mathrm{R})$$

and the right-hand side can be written

$$(m_\mathrm{L}t't'_\mathrm{R} + t_\mathrm{L}m't'_\mathrm{R} + t_\mathrm{L}t'm'_\mathrm{R}, t_\mathrm{L}t't'_\mathrm{R}).$$

Using the relationships $tt'_\mathrm{R} = t_\mathrm{R} = t't'_\mathrm{R}$ and $t_\mathrm{L}t = t'_\mathrm{L} = t_\mathrm{L}t'$, equation (6.7) can be established by showing

(6.9) $$m_\mathrm{L}t_\mathrm{R} + t_\mathrm{L}mt'_\mathrm{R} + t'_\mathrm{L}m'_\mathrm{R} = m_\mathrm{L}t_\mathrm{R} + t_\mathrm{L}m't'_\mathrm{R} + t'_\mathrm{L}m'_\mathrm{R}.$$

But $(m_\mathrm{L}, n_\mathrm{L})$ and $(m'_\mathrm{R}, n'_\mathrm{R})$ belong to $\#\varphi$, and since φ is compatible with the actions of T on M and N, we have $(m_\mathrm{L}t_\mathrm{R}, n_\mathrm{L}t_\mathrm{R}) \in \#\varphi$ and $(t'_\mathrm{L}m'_\mathrm{R}, t'_\mathrm{L}n'_\mathrm{R}) \in \#\varphi$. Therefore, (6.9) is implied by (6.8), and

$$K_{\varphi ** T} < K_\varphi.$$

The assertion is proven. □

A derived category version of Theorem 6.2 is also valid, but it was not discovered in time for inclusion in [8]. For completeness, this subject is discussed briefly in Appendix B.

Proposition 6.3. *Let $V ** T$ be a double semidirect product, and let $\pi : V ** T \to T$ be the projection morphism. Then*

$$K_\pi \sim V.$$

Proof. The projection $\pi : V ** T \to T$ is easily seen to coincide with the morphism

$$\varphi ** T : V ** T \to 1 ** T$$

where $\varphi : V \to 1$ is the collapsing morphism. By Theorem 6.2 and by Proposition 1.3, we obtain

$$K_\pi \approx K_{\varphi ** T} \sim K_\varphi \approx V. \qquad \square$$

Proposition 6.4. *Let $\theta : M \to V**N$ be an injective relation of monoids, and let $\varphi = \theta\pi : M \to N$ be the relation obtained by composing θ with the projection. Then*

$$K_\varphi \prec V.$$

Proof. Consider the commuting diagram of monoid relations

$$\begin{array}{ccc} M & \xrightarrow{\ \theta\ } & V**N \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & N & \end{array}$$

Since $\theta : M \to V**N$ is injective, we may apply Proposition 5.1 to obtain $K_\varphi \prec K_\pi$. Proposition 6.3 then yields the assertion. $\square$

7. The block product

We now define the block product of two monoids; the block product is a particular double semidirect product. The block product contains, as we shall see, all double semidirect products as submonoids. This product stands in the same relationship to the double semidirect product as the wreath product does to the semidirect product.

Let V and T be monoids. We will write V additively with identity 0, although commutivity for V is not assumed. By $V^{T\times T}$ we mean the monoid of all functions from $T\times T$ to V equipped with coordinatewise multiplication. $V^{T\times T}$ will also be denoted additively; the identity is the function f_0, whose value is always 0. $V^{T\times T}$ is isomorphic to a direct product of card $T\times T$ copies of V.

There is a natural double action of T on $V^{T\times T}$ defined as follows:

$$T\times V^{T\times T}\times T \to V^{T\times T},$$

$$(t, f, t') \to [t, f, t'], \tag{7.1}$$

$$t_{\mathrm{L}}[t, f, t']t_{\mathrm{R}} = (t_{\mathrm{L}}t)f(t't_{\mathrm{R}}).$$

The notation $[t, f, t']$ is used to denote the result of the double action. This is done to avoid confusion with the notation tft', which is the result of evaluating f at (t, t'). Thus, $[t, f, t'] \in V^{T\times T}$, while $tft' \in V$.

The reader may readily verify that the action (7.1) satisfies conditions (6.2). These conditions, translated to this setting, are

$$(7.2) \quad \begin{aligned} &[t, f_1+f_2, t'] = [t, f_1, t'] + [t, f_2, t'], \qquad [t_1, [t_2, f, t_2'], t_1'] = [t_1 t_2, f, t_2' t_1'],\\ &[1, f, 1] = f, \qquad [t, f_0, t'] = f_0 \end{aligned}$$

for all $t, t', t_1, t_2, t_1', t_2' \in T$ and $f, f_1, f_2 \in V^{T\times T}$.

The *block product* $V \square T$ of monoids V and T is the double semidirect product $V^{T\times T} ** T$ associated with the double action (7.1). Therefore, $V \square T$ is the monoid with underlying set $V^{T\times T} \times T$ and product

$$(f, t)(f', t') = ([1, f, t'] + [t, f', 1], tt').$$

We now verify our claim that $V \square T$ contains every double semidirect product $V ** T$. For any given double semidirect product $V ** T$, define the function

$$\theta : V ** T \to V \square T, \qquad (v, t)\theta = (f_v, t), \qquad t_1 f_v t_2 = t_1 v t_2.$$

Since $1 f_v 1 = 1 v 1 = v$, we see that θ is an injective function. Furthermore, θ maps the identity $(0, 1)$ to the identity $(f_0, 1)$ of $V \square T$. Lastly,

$$\begin{aligned} (v, t)\theta(v', t')\theta &= (f_v, t)(f_{v'}, t') = ([1, f_v, t'] + [t, f_{v'}, 1], tt')\\ &= (f_{vt' + tv'}, tt') = ((v, t)(v', t'))\theta. \end{aligned}$$

These observations combine to prove the following proposition:

Proposition 7.1. *Every double semidirect product $V ** T$ is (isomorphic to) a submonoid of $V \square T$.* □

We list additional properties of the block product. The first three are inherited from corresponding properties, (6.3)–(6.5), of the double semidirect product.

$$(7.3) \qquad 1 \square T \approx T \quad \text{and} \quad V \square 1 \approx V.$$

$$(7.4) \qquad V \prec V \square T \quad \text{and} \quad T \prec V \square T.$$

$$(7.5) \qquad V^{\varrho} \square T^{\varrho} \approx (V \square T)^{\varrho}.$$

The remaining properties, listed below, are followed by brief indications of their proofs.

$$(7.6) \qquad V \prec V', \ T \prec T' \quad \Rightarrow \quad V \square T \prec V' \square T'.$$

$$(7.7) \qquad V \circ T \prec V \square T.$$

$$(7.8) \qquad V \circ_{\varrho} T \prec V \square T.$$

For (7.6), let $\varphi : V \prec V'$ and $\theta : T \prec T'$ be the given divisions. Define a relation $\psi : V \square T \to V' \square T'$ by the rule

$$(f, t)\psi = \{(h, t') \in V' \square T' : t' \in t\theta, \ t_L' h t_R' \in (t_L f t_R)\varphi \ \ \forall t_L' \in t_L\theta, \ t_R' \in t_R\theta\}.$$

A routine calculation shows that ψ is a division.

For (7.7), define

$$\theta: V \circ T \to V \square T, \qquad (f,t)\theta = (\bar{f}, t), \qquad t\bar{f}t' = tf$$

and show that θ is an injective morphism. A dual argument proves (7.8). Alternately, (7.8) may be established by using (7.7), the isomorphism $V \circ_{\varrho} T \approx (V^{\varrho} \circ T^{\varrho})^{\varrho}$ and (7.5). For then

$$V \circ_{\varrho} T \approx (V^{\varrho} \circ T^{\varrho})^{\varrho} < (V^{\varrho} \square T^{\varrho})^{\varrho} \approx V \square T.$$

Let $\varphi: M \to N$ be a relation of monoids and let T be a monoid. Then φ induces a relation of monoids

(7.9) $$\varphi \square T: M \square T \to N \square T, \qquad (f,t)\varphi \square T = \{(h,t): t_{\mathrm{L}} h t_{\mathrm{R}} \in (t_{\mathrm{L}} f t_{\mathrm{R}})\varphi\}$$

as can be verified by direct computation.

Proposition 7.2. *Let $\varphi: M \to N$ be a relation of monoids, and let T be a monoid. Then*

$$K_{\varphi \square T} \sim (K_{\varphi})^{T \times T}.$$

Proof. The relation $\varphi: M \to N$ induces the product relation

$$\varphi^{T \times T}: M^{T \times T} \to N^{T \times T}, \qquad f\varphi^{T \times T} = \{h \in N^{T \times T}: tht' \in (tft')\varphi\}.$$

We claim that $\varphi^{T \times T}$ is compatible with the double actions (7.1) of T on $M^{T \times T}$ and $N^{T \times T}$ used to define the block product. To show this, let $(f,h) \in \#\varphi^{T \times T}$. We need to verify that

$$([t_{\mathrm{L}}, f, t_{\mathrm{R}}], [t_{\mathrm{L}}, h, t_{\mathrm{R}}]) \in \#\varphi^{T \times T}$$

for all $t_{\mathrm{L}}, t_{\mathrm{R}} \in T$. This can be seen by the calculation

$$t[t_{\mathrm{L}}, h, t_{\mathrm{R}}]t' = (tt_{\mathrm{L}})h(t_{\mathrm{R}}t') \in ((tt_{\mathrm{L}})f(t_{\mathrm{R}}t'))\varphi = (t[t_{\mathrm{L}}, f, t_{\mathrm{R}}]t')\varphi.$$

Therefore, $\varphi^{T \times T}$ induces a relation of monoids

$$(\varphi^{T \times T}) ** T: M^{T \times T} ** T \to N^{T \times T} ** T$$

as defined in Section 6. This is exactly relation (7.9); that is to say, $(\varphi^{T \times T}) ** T = \varphi \square T$.

We now apply Theorem 6.2 to obtain the assertion. Theorem 6.2 states that

$$K_{\varphi \square T} \sim K_{(\varphi^{T \times T})}.$$

However, by Proposition 1.7,

$$K_{(\varphi^{T \times T})} \approx (K_{\varphi})^{T \times T}.$$

The assertion follows from these two statements. $\square$

Since $V \square T$ is a double semidirect product, we may speak of the projection morphism

$$\pi : V \square T \to T, \qquad (f, t)\pi = t.$$

This projection may also be described as the induced morphism

$$\varphi \square T : V \square T \to 1 \square T \approx T$$

where $\varphi : V \to 1$ is the collapsing morphism. Since $K_\varphi \approx V$ by Proposition 1.3, this description of the projection leads to the following result:

Proposition 7.3. *Let $\pi : V \square T \to T$ be the projection morphism. Then*

$$K_\pi \sim V^{T \times T}. \qquad \square$$

Proposition 7.3 can also be established by noting that $V \square T$ is a double semidirect product $V^{T\times T} ** T$, and appealing to Propositions 6.3.

The following theorem states the principal relationships between the kernel and the block product:

Theorem 7.4 (Kernel Theorem). (a) *Let $\varphi : M \to N$ be a relation of monoids, and let V be a monoid satisfying $K_\varphi < V$. Then there is an injective relation of monoids*

$$\theta : M \to V \square N$$

satisfying $\theta\pi = \varphi$. In particular, if φ is fully defined, then θ is a division.

(b) *Let $\theta : M \to V \square N$ be an injective relation of monoids, and let*

$$\varphi = \theta\pi : M \to N$$

be the relation obtained by composing θ with the projection. Then

$$K_\varphi < V^{N \times N}.$$

Proof. (a) The last statement follows from the first because θ and φ have the same domain. Since φ is fully defined, θ is fully defined and injective, that is, θ is a division. We therefore prove the first statement.

Let $\psi : K_\varphi < V$ be the given division. For each pair $(m, n) \in \# \varphi$, define the set of functions

$$F(m,n) = \{ f \in V^{N \times N} : \ n_1 f n_2 \in [n_1, (m,n), n_2]\psi, \ n_1, n_2 \in M\varphi \}.$$

Note that $F(m,n)$ is always non-empty and that $f_0 \in F(1,1)$.

Define the relation

$$\theta : M \to V \square N, \qquad m\theta = \{ (f, n) : \ n \in m\varphi \text{ and } f \in F(m,n) \}.$$

Because $F(m,n) \neq \emptyset$, it follows easily that $\varphi = \theta\pi$. We will show that θ is an injective relation of monoids.

We first show that θ is a relation of monoids. Since $f_0 \in F(1,1)$, we see that $(f_0, 1) \in 1\theta$. Let $(f, n) \in m\theta$ and $(f', n') \in m'\theta$, and consider the product

$$(f,n)(f',n') = ([1,f,n'] + [n,f',1], nn').$$

Now $f \in F(m,n)$ and $f' \in F(m',n')$, so for every $n_1, n_2 \in M\varphi$ we have

$$\begin{aligned} n_1([1,f,n'] + [n,f',1])n_2 &= n_1 f(n'n_2) + (n_1 n)f'n_2 \\ &\in [n_1,(m,n),n'n_2]\psi + [n_1 n,(m',n'),n_2]\psi \\ &\subseteq [n_1,(mm',nn'),n_2]\psi. \end{aligned}$$

The last inclusion follows because $[n_1,(m,n),n'n_2]$ and $[n_1 n,(m',n'),n_2]$ are consecutive arrows of K_φ. It follows that

$$[1,f,n'] + [n,f',1] \in F(mm',nn')$$

and

$$(f,n)(f',n') \in (mm')\theta.$$

Therefore, $\theta : M \to V \square N$ is a relation of monoids.

It remains to show that θ is injective. Let (f,n) belong to both $m\theta$ and $m'\theta$. Then for each $n_1, n_2 \in M\varphi$, $n_1 f n_2$ is an element of both $[n_1,(m,n),n_2]\psi$ and $[n_1,(m',n),n_2]\psi$. However, ψ is injective on the hom-sets of K_φ, so we conclude that

$$[n_1,(m,n),n_2] = [n_1,(m',n),n_2] \quad \forall n_1, n_2 \in M\varphi.$$

In particular, this equation holds when $n_1 = 1 = n_2$. Therefore by Lemma 1.1(a), $m = m'$. This shows that θ is injective and establishes part (a) of the theorem.

(b) Let $\theta : M \to V \square N$ be an injective relation of monoids, and consider the commuting diagram

$$\begin{array}{ccc} M & \xrightarrow{\theta} & V \square N \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & N. & \end{array}$$

Since θ is injective, we may apply Proposition 5.1 to obtain $K_\varphi < K_\pi$. Proposition 7.3 then yields the assertion. □

We wish to show here that in part (a) of the Kernel Theorem, the block product cannot be replaced by a double semidirect product. For consider a surjective morphism $\varphi : G \to H$, where G and H are finite groups. By Proposition 1.4, $K_\varphi \sim \ker \varphi$. If some double semidirect product sufficed for Theorem 7.4(a), then we could write

$$G < \ker \varphi ** H.$$

But both these groups have the same cardinality, so they must be isomorphic. Furthermore, by Example 6.1, $\ker \varphi ** H$ is isomorphic to a semidirect product $\ker \varphi * H$. This shows that G is a split extension. But not all groups are split extensions, for example, $\mathbb{Z}_4$. Therefore, the block product cannot be replaced by a double semidirect product in the Kernel Theorem.

We may now apply the Kernel Theorem to relations of finite monoids.

Corollary 7.5. *Let Q be a finite simple monoid and let $\varphi : M \rightarrow N$ be a finite, fully defined relation that is either locally trivial or primitive of type Q. Then for some finite monoid $V \in (Q)$, we have*

$$M \prec V \square N.$$

Proof. If φ is primitive of type Q, then $K_\varphi \in (Q)$. If φ is locally trivial, then $K_\varphi \in \mathcal{A}$. But $\mathcal{A} \subseteq (Q)$ by [8, Theorem 8.1], so again $K_\varphi \in (Q)$. Thus in both cases, $K_\varphi \prec V$ for some finite monoid $V \in (Q)$. The assertion now follows from part (a) of Theorem 7.4. □

Proposition 7.6. (a) *Let $\varphi : M \rightarrow N$ be a finite, fully defined relation that is aperiodic. Then there exists a division*

$$M \prec V_k \square (\cdots \square (V_2 \square (V_1 \square N))\cdots)$$

where $V_1, \ldots, V_k$, $k \geq 0$, are finite monoids in (U_1).

(b) *Let $\varphi : M \rightarrow N$ be a finite, fully defined relation that is U_1-free. Then there exists a division*

$$M \prec V_k \square (\cdots \square (V_2 \square (V_1 \square N))\cdots)$$

where $V_1, \ldots, V_k$, $k \geq 0$, are finite groups.

(c) *Let $\varphi : M \rightarrow N$ be a finite, fully defined relation that is both aperiodic and U_1-free. Then for any simple monoid Q, there exists a division*

$$M \prec V_k \square (\cdots \square (V_2 \square (V_1 \square N))\cdots)$$

where $V_1, \ldots, V_k$, $k \geq 0$, are finite monoids in (Q).

Proof. (a) Let $\varphi : M \rightarrow N$ be an aperiodic, fully defined relation. Then by Corollary 3.4, φ has a decomposition $\varphi = \varphi_k \ldots \varphi_1$, where each factor φ_i is either locally trivial or is primitive of type U_1. Therefore, as in the proof of Corollary 7.5, the kernel of each factor belongs to (U_1). Furthermore, since φ is fully defined, each factor φ_i is fully defined. This allows us to iteratively apply Corollary 7.5.

Let $\varphi_k : M \rightarrow M_k$. Then by Corollary 7.5 we have $M \prec V_k \square M_k$, for some monoid $V_k \in (U_1)$. Let $\varphi_{k-1} : M_k \rightarrow M_{k-1}$. Then $M_k \prec V_{k-1} \square M_{k-1}$. Using (7.6), we may write

$$M \prec V_k \square (V_{k-1} \square M_{k-1}).$$

The assertion of (a) now follows by iteration. Parts (b) and (c) are proved in a similar manner. □

The monoids $V_1, \ldots, V_k$ of Proposition 7.6 cannot be specified in advance. Only the varieties they belong to can be predicted. This means that the proper setting for

Proposition 7.6 is varieties, rather than individual monoids. Part 2 of this paper, mentioned in the introduction, treats the results of this paper in a variety setting.

Appendix A. Finite semigroups and monoids

Properties of finite semigroups that are not valid in the infinite case are used throughout the paper without much comment. These facts are briefly outlined below.

(1) Every element of a finite semigroup, when raised to a sufficiently high power, is an idempotent.

(2) Let M be a monoid. If for each $m \in M$, $m^k = 1$ for some $k \geq 1$, then M is a group.

(3) Every monoid that is not a group contains a copy of the simple monoid $U_1 = \{1, 0\}$. For if M is not a group, then by (2) there exists an element $m \in M$ and some $k \geq 1$ such that m^k is an idempotent not equal to 1. Then $\{1, m^k\} \approx U_1$.

(4) Let $\varphi : S \to T$ be a morphism, and let T' be a monoid (group) in T. Let S' be a subsemigroup of S of the smallest possible cardinality satisfying $S'\varphi = T'$. Then S' is a monoid (group) in S.

(5) Let $a, b \in S$. If $a \geq b$ in the $\mathscr{R}$-ordering of S and $a \mathscr{J} b$, then $a \mathscr{R} b$. Dually, if $a \geq b$ in the $\mathscr{L}$-ordering of S and $a \mathscr{J} b$, then $a \mathscr{L} b$.

(6) Let G be a maximal group in S with idempotent e, and let J be the $\mathscr{J}$-class of S containing G. The $eSe \cap J = G$.

(7) Let M be a monoid with maximal subgroup G. Then $M - G$ is an ideal of M. For let J be the $\mathscr{J}$-class of M containing G. Then every element of M is either in J or is strictly below J in the $\mathscr{J}$-class ordering. Thus $M - J$ is an ideal. But by applying (6) with $e = 1$, we obtain $J = M \cap J = G$. Therefore, $M - G$ is an ideal of M.

Proofs of these facts can be found throughout the literature. For example (1)–(4) can be found in [2], while (5) and (6) appear in [7]. See also [1].

Appendix B. The derived category and the semidirect product

Theorem 6.2 states a fundamental relationship between the kernel and the double semidirect product. The derived category and the semidirect product enjoy a similar relationship.

Theorem B.1. *Let $\varphi : M \to N$ be a relation of monoids, and let T be a monoid. Suppose there are left actions of T on M and T on N that are compatible with φ. Then*

$$D_{\varphi * T} \sim D_{\varphi}. \qquad \square$$

The notions of compatible action and the definition of

$$\varphi * T : M * T \to N * T$$

are exactly as defined in Section 6. The proof of Theorem B.1 can be obtained by appropriately simplifying the proof of Theorem 6.2. Analogs of Propositions 6.3 and 6.4 are then easily obtained.

Proposition B.2. *Let $V*N$ be a semidirect product, and let $\pi : V*N \to N$ be the projection morphism. Then $D_\pi \sim V$.* □

Proposition B.3. *Let $\theta : M \to V*N$ be an injective relation of monoids, and let $\varphi = \theta\pi : M \to N$ be the relation obtained by composing θ with the projection. Then $D_\varphi < V$.* □

Theorem B.1 was discovered too late to be included in [8]. However, key results in [8] can now be seen as corollaries of Theorem B.1. We briefly discuss these results.

First, we may define the wreath product version of (7.9). Let $\varphi : M \to N$ be a relation of monoids, and let T be a monoid. Then φ induces a relation of monoids

(B.1) $$\varphi \circ T : M \circ T \to N \circ T, \qquad (f, t)\varphi \circ T = \{(h, t) : \ t_0 h \in (t_0 f)\varphi \ \ \forall t_0 \in T\}$$

as can be verified by direct computation. Then, following the proof of Proposition 7.3 and using Theorem B.1 in place of Theorem 6.2 leads us to the following result:

Proposition B.4. *Let $\varphi : M \to N$ be a relation of monoids, and let T be a monoid. Then*

$$D_{\varphi \circ T} \sim (D_\varphi)^T. \qquad \square$$

Propositions 7.5 and 7.6 are corollaries of Proposition 7.3. The corresponding results for the wreath product and the derived category appear in [8]. They are

Proposition B.5 (Proposition 5.1 of [8]). *Let $\pi : V \circ N \to N$ be the projection morphism. Then $D_\pi \sim V^N$.* □

Proposition B.6 (Theorem 5.2(b) of [8]). *Let $\theta : M \to V \circ N$ be an injective relation of monoids, and let $\varphi = \theta\pi : M \to N$ be the relation obtained by composing θ with the projection. Then $D_\varphi < V^N$.* □

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